

ARCTAN FORMULA FOR PI

For many years I have been studying various integral versions of the $\arctan(1/N)$ function and its role in determining the value of π . The standard starting points for such an analysis are the integrals-

$$\arctan\left(\frac{1}{N}\right) = N \int_{x=0}^{x=1} \frac{dx}{N^2 + x^2} = \frac{\Delta^2}{N} \int_{t=0}^{\infty} \frac{dt}{\sqrt{(\Delta^2 + t^2)(1 + t^2)(\Delta^2 + t^2)}}$$

where $\Delta = N/\sqrt{N^2+1}$ is a new parameter with the range $0 < \Delta < 1$ and $x = t/\sqrt{1+t^2}$. Expanding the first of these integrals, using the geometric series, produces the slowly convergent series-

$$\arctan\left(\frac{1}{N}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)N^{2n+1}}$$

The usual way to speed up the convergence of this series is to replace $\arctan(1/N)$ by a set of \arctan terms where the individual $\arctan(1/M)$ s all have $M \gg N$. One of the better of these formulas is our own expansion-

$$\arctan(1) = \pi/4 = 12\arctan(1/38) + 20\arctan(1/57) + 7\arctan(1/239) + 24\arctan(1/268)$$

This four term equality has all four of its \arctan terms positive, the numerator term in the \arctan is always equal to unity, and $M \geq 38$. We have used an on-line calculator to verify this result to over 1000 decimal places.

The disadvantage of multiple \arctan formulas is the need to sum multiple sums. To get around this difficulty one could try to develop a one term \arctan formula. We want to show here how this can be done in general and what limitations there are to such an approach.

Start with the identity-

$$2 \arctan\left(\frac{1}{N}\right) = \arctan\left[\frac{2N}{N^2 - 1}\right]$$

which is equivalent to saying-

$$\arctan\left(\frac{1}{a}\right) = 2\arctan\left[\frac{1}{(a + \sqrt{a^2 + 1})}\right]$$

where $a = 2N/(N^2 - 1)$. If now $a_0 = \sqrt{3}$, we find $\pi/6 = 2\arctan[1/(2 + \sqrt{3})]$. Thus-

$$\frac{\pi}{12} = \arctan\left(\frac{1}{2 + \sqrt{3}}\right)$$

Next letting $a_1=2+\sqrt{3}$, we find-

$$\frac{\pi}{24} = \arctan \left[\frac{1}{2 + \sqrt{3} + \sqrt{(2 + \sqrt{3})^2 + 1}} \right]$$

The trend is obvious, namely, that-

$$\pi = 6 \cdot 2^n \cdot a_n \int_{x=0}^1 \frac{dx}{(a_n^2 + x^2)} \quad \text{with} \quad a_{n+1} = a_n + \sqrt{a_n^2 + 1} \quad \text{and} \quad a_0 = \sqrt{3}$$

Expanding this integral in powers of the variable $(x/a_n)^2 \ll 1$, one finds the series-

$$\pi = \frac{6 \cdot 2^n}{a_n} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)a_n^{2k}}$$

The drawback of this type of expansion for π is that a_n will no longer be an integer hence complicating the evaluation procedure. However in theory at least this last series represents an exact expression for π .

Let us next calculate the values of a_n using the one line MAPLE program-

```
a[0]:=sqrt(3); a[1]:=2+sqrt(3); for n from 1 to 9 do a[n+1]:=evalf(sqrt(a[n]^2+1)+a[n],90)od;
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Here are the results-

```
a[0] := sqrt( 3)
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a[1] := 2 + sqrt(3)
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a[2] := 7.59575411272515044052641914042146183747842460984399882966\
517928469362587282822210247139316
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a[3] := 15.2570516882655391096173820057228664698790442110909778599\
606686239314541764529616643838146
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a[4] := 30.5468399869440508056740229603391925634143485168851969908\
976061633085605592561557282731711
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a[5] := 61.1100438960234131940133869992913550717083005807677160746\
322759169662950960915844709731561
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a[6] := 122.228269205450620264397150276743892216031231130258162890\
381027027387694798477168131092424

a[7] := 244.460629049152027884964565488853226881407317823371669830\
995882596064538358181256987028593

a[8] := 488.923303408873275513517668229591098461279744749013270018\
173211863290379587996400921771326

a[9] := 977.847629471961645868190276557208118696021673299627216578\
349438805811925899398670840008518

a[10] := 1955.6957702708971499878508571522253280415396819739805141\
7517640847262832561563096454845617

Notice that we have terminated the values of a_n at 90 digits so that one can expect accuracy for π to not be better than this number of digits. If we take $a[10]$ which is close in value to 1956 we can stop the series at-

$$\log[(2k+1)(1956)^{2k}] = \log(2k+1) + 2k \log(1956) \approx 90$$

where log means the base 10 logarithm. Solving we find $k < 14$. Thus a ninety digit accurate value for π is given by -

$$\text{evalf}(12 \cdot 2^{10} / a[10] \cdot \sum_{k=0..14} ((-1)^k / (a[10]^{2k} \cdot (2k+1))), 90);$$

which produces-

$\pi =$
3.1415926535897932384626433832795028841971693993751058209749445923078164062862
0899862803483...

This result is indeed good to 90 decimal places. Whenever I generate numbers like this with very little effort, I think back to Ludolph van Ceulen (1539-1610) of Holland who spent essentially his entire life obtaining just the first 35 places of π using the cumbersome Archimedes Method of Regular polygons. Van Ceulen was so proud of his efforts that he had the digits corresponding to the 33rd, 34th, and 35th digit engraved on his tombstone. In Germany the number π is still often referred to as the Ludolph Number. There are many mnemonics available for memorizing π . One of these is "drei komma Hus verbrannt". I remember Karl Pholhausen (of boundary layer fame) mentioning this mnemonic to me while he was a visiting professor here at the University

of Florida many years ago. He apparently learned it back in Germany as a child. Translating it says 3+decimal point+1415 , with 1415 being the year the Bohemian religious reformer Jan Hus(1369-1415) was burned(verbrannt) at the stake in Constance for heresy. Another is the Otto ratio $355/113=3.141592\dots$ good to six places and easy to remember by involving each of the first three odd numbers 1,3,5 twice. Then there is also the long mnemonic-

3 1 4 1 5 9 2 6 5 3 5 8 9 7 9

How I like a drink, alcoholic of course, after the heavy lectures involving quantum mechanics.

This 14 digit mnemonic is based on the number of letters in a word as indicated in blue . It has the shortcoming that it cannot handle the digit 0. Fortunately the first zero in π does not occur until the 32nd digit to the right of the decimal point.

Another manipulation one can do is to find the arctan(1/N) where N is a number close to that produced by the above a_{n+1} manipulation. Consider the slightly different iteration based on an $a_0=1$ where $\arctan(1/a_0)=\pi/4$. On doing these iterations for a_0 based on the same iteration formula $a_{n+1}=a_n+\sqrt{a_n^2+1}$ we find-

$$a[0] := 1$$

$$a[1] := 1 + \sqrt{2}$$

$$a[2] := 5.02733949212584810451497507106407238573719425207548712827$$

$$a[3] := 10.1531703876088604621071476634194722037674409548501769525$$

$$a[4] := 20.3554676249871881783196386481102580246145776612527485918$$

$$a[5] := 40.7354838720833018007438570501814247704216165212224370708$$

Thus looking at a_5 (here given to 58 decimal places) we have that-

$$\arctan\left[\frac{1}{38}\right] = \frac{\pi}{4 \cdot 2^5} + \arctan\left\{\frac{a_5 - 38}{38a_5 + 1}\right\} =$$

$$0.024543692606170259675489401431871116282790385932618014227 +$$

$$0.001766024646751932454419329529466103516960756543629856217 =$$

$$0.02630971725292219212990873096133721979975114247624787044$$

which gives the correct value of arctan(1/38) to an accuracy of 56 decimal places. Notice the small second term. This follows from the fact that a_5-38 is a small number while $a_5 \times 38$ is large.

Let us try a similar approach with the well known Machin formula-

$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

Looking at $a[2] := 5.0273394921\dots$ where $\arctan(1/a_2) = \pi/16$, we can rewrite the Machin Formula as-

$$\begin{aligned} \arctan\left(\frac{1}{239}\right) &= 4 \arctan\left(\frac{a[2] - 5}{5a[2] + 1}\right) \\ &= 0.0041840760020747238645382149592854527410480653076319 \end{aligned}$$

good to 48 places.

Again we see that excellent approximations to $\arctan(1/N)$ can be generated by choosing the right a_0 whose n th iteration lies near N . It makes for a relatively rapid evaluation of $\arctan(1/N)$ since –

$$a_n \int_0^1 \frac{dx}{a_n^2 + x^2} = \arctan\left(\frac{1}{a_n}\right) = \frac{\pi}{b2^n} \quad \text{with} \quad \arctan\left(\frac{1}{a_0}\right) = \frac{\pi}{b}$$

and one does not have to integrate the integral explicitly. For $a_0=1$ we have $b=4$ and for $a_0=\sqrt{3}$ we find $b=6$. Expanding the integral in a series in $(x/a_n)^2$ works but can become rather tedious when actually evaluating such a series because of the long non-integer form of the a_n .

Finally let us look at another identity for π based on the $\arctan(1/N)$. Consider the following integral and its expansion-

$$\int_{x=0}^1 \frac{dx}{(1+x^2)(N^2+x^2)} = \frac{\pi}{4(N^2-1)} - \frac{1}{N(N^2-1)} \arctan\left(\frac{1}{N}\right)$$

This is equivalent to saying-

$$\pi = \frac{4}{N} \arctan\left(\frac{1}{N}\right) + 4(N^2-1) \int_{x=0}^1 \frac{dx}{(1+x^2)(N^2+x^2)}$$

If $N=5$ we get-

$$\pi = 4/5) \arctan(1/5) + 96 \int_0^1 \frac{dx}{(1+x^2)(25+x^2)} =$$

0.157916447879904606696039812156 + 2.98367620570988863176660357112 =

3.14159265358979323846264338328... We have attempted to solve the last integral by AGM methods but so far have been unsuccessful. The arithmetic and

geometric mean of 1 and 5 is $M=3.032\dots$. It does not yield a correct answer for the equivalent integral even after the transformation $x=t/\sqrt{t^2+1}$ is introduced.

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