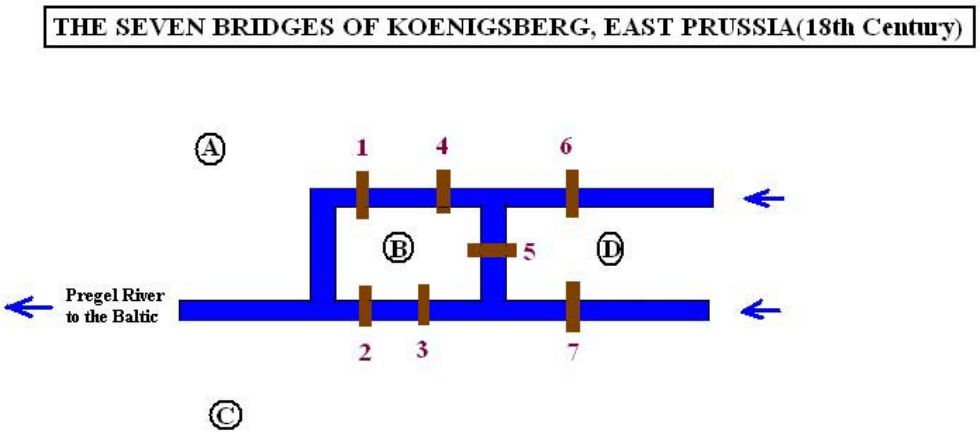
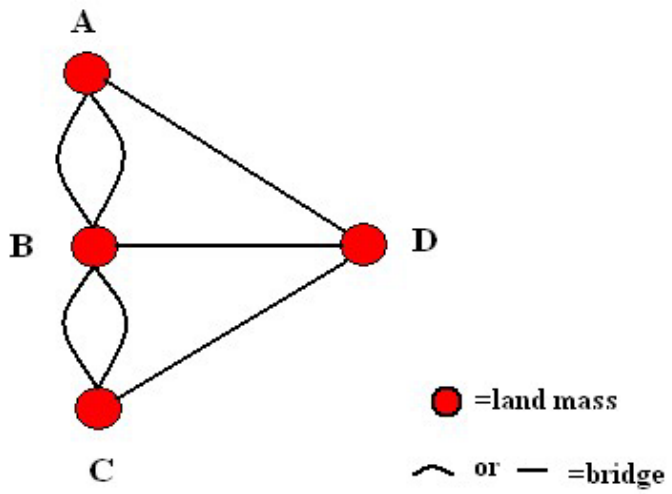
THE SEVEN BRIDGES OF KOENIGSBERG AND RELATED PROBLEMS

In the city of Koenigsberg, East Prussia (now called Kaliningrad and famous for its university whose faculty included Immanual Kant, Hermann von Helmholtz, and Friedrich Bessel) there once existed seven bridges which connected different parts of the town as shown –



A question which locals would pose to visitors during the 17 hundreds was "Is it possible to cross all bridges just once and end up at the same starting point?. Leaonard Euler, who was working in St. Petersburg at the time, heard about this query and showed (via the invention of Graph Theory) that such a circuit is impossible. Here is his argument. Consider the shown land masses A, B, C, and D as vertexes represented by circles and then draw curves between the vertexes representing each bridge. Unless all but two vertexes show an even number of curves passing to a vertex no Euler Path with different starting and ending points crossing all bridges is possible. Also unless the number of curves (edges) at each vertex is even, a complete Euler Circuit with the same starting and ending point is impossible. The Euler Circuit will in general be more difficult to achieve than an Euler Path. For the Koenigsberg Bridge Problem one has the following graph-

GRAPH THEORY REPRESENTATION OF KOENIGSBERG BRIDGE PROBLEM

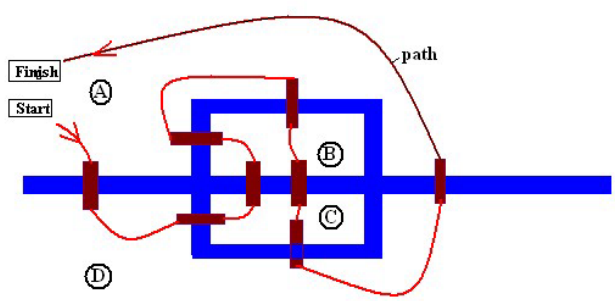


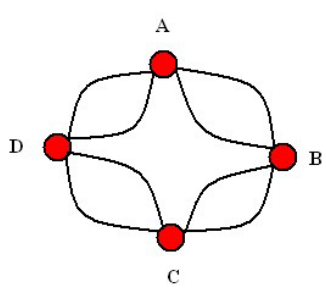
No Solution for Euler Path or Euler Circuit possible since A, B, C, and D all have an odd number of curves touching

Note that all four vertexes have an odd number of bridges connecting. Thus neither an Euler Circuit or Euler Path is possible.

Lets next try some related problems such as the following-

EIGHT BRIDGE PROBLEM SHOWING COMPLETE EULER CIRCUIT

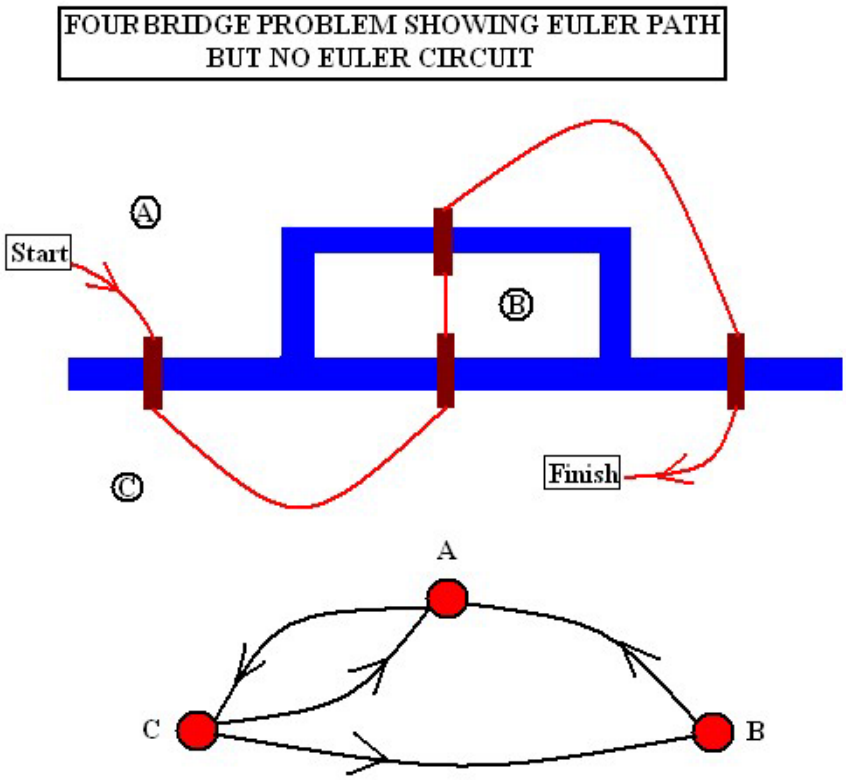




An even number of edges touching each vertex. Hence Euler Circuit possible.

Here we have eight bridges and one can draw a circuit crossing each bridge just once and still ending up at the same finish point as the start. The Euler Graph shown indicates four edges touch each vertex. Hence we have not only an Euler Path but also a complete Euler Circuit as shown.

As another example consider the four bridge problem -



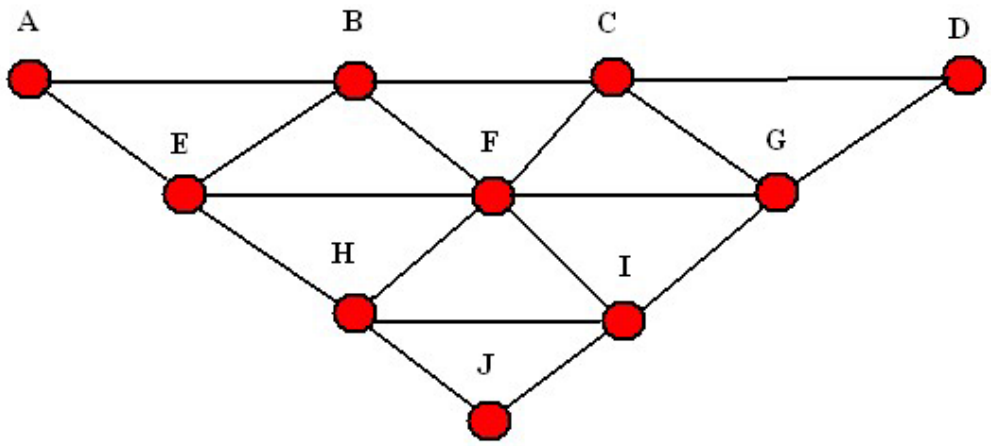
Two odd vertex touchings plus one even touching means Euler Path possible but no complete Euler Circuit

Here one can cross each bridge just once in order to cross them all. However, a complete Euler Circuit is impossible since the start and finish points end up in different land areas. According to Graph Theory, not all vertexes have an even number of edges touching them, and hence, according to Euler, no Euler Circuit can exist. The fact that we have only two vertexes which are touched by an odd number of edges in this example, means that an Euler Path exits.

There are many other problems which may be treated by Graph Theory. One of these is the Tourist problem N cities are connected by M roads and one asks the question is it possible to visit all the cities using all the roads but never moving along on the same road twice? The answer follows from Euler's Koenigsberg Bridge Problem. To demonstrate, look at the following-

TOURIST PROBLEM

(Starting at city A is it possible to visit all remaining 9 cities by never re-travelling any of the 18 connecting roads before ending again at A?)

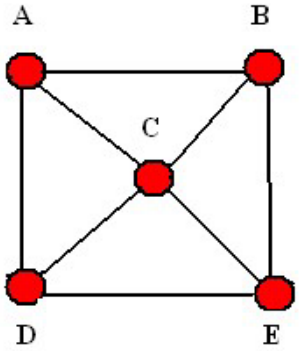


Answer: YES

Among several routes A-E-B-C-D-G-C-F-G-I-F-H-I-J-H-E-F-B-A is a posibility.

Here the red dots represent the cities (called vertexes in Graph Theory) and the black lines are the interconnecting roads (called edges in Graph Theory). We note that seven of the ten vertexes have four edges touching and the remaining three vertexes have two edges touching. Thus all ten vertexes have an even number of edges touching and thus one is guaranteed a complete Euler Circuit. There are many such possible routes the tourist might take and cover all roads. A possible route is A-B-C-D-G-C-F-B-E-F-G-I-F-H-I-J-H-E-A. Another is A-E-B-C-D-G-C-F-G-I-F-H-I-J-H-E-F-B-A. To visit each city just once starting at A one can do so without roads AE, EB, BF, FC, CG,GI, FI, HF,and HJ. No Euler Circuit or Euler Path would be possible for a five city arrangement with eight roads connecting as shown-

FIVE CITY PROBLEM WITH EIGHT CONNECTING ROADS



Vertexes A, B, D, and E have 3 edges each, Vertex C has 4 edges
Hence no Euler Circuit or Euler Path

Another mathematical problems which can be handled by Graph Theory is the Four-Color Map Problem.

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