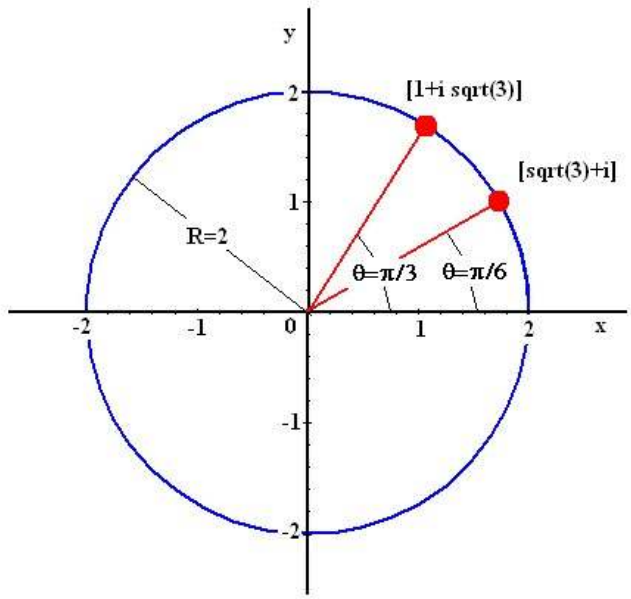
EXAMINATION OF THE COMPLEX NUMBER
N=(a+ib)^n=(b+ia)^n

Under certain conditions the complex number a+ib taken to the nth integer power will be equal to b+ia taken to the same power. This observation follows from the fact that the magnitude of a+ib and b+ia are both equal to sqrt(a^2+b^2) and hence the two numbers lie on the same circle within the complex plane. If one multiplies each of these numbers by themselves n times it is possible that the product ends up at the same point on the (a^2+b^2)^(n/2) radius circle so that (a+ib)^n=(b+ia)^n. Let us demonstrate this point by considering the numbers sqrt(3)+i and 1+i sqrt(3). They lie on a radius 2 circle in the complex plane as shown-



If we now multiply [1+isqrt(3)] twelve times by itself the new number equals 2^12. Next multiplying [sqrt(3)+i)] six times by itself will yield 2^6 and twelve times by itself will yield 2^12. Thus we have the identity-

$$[\sqrt{3}+i]^{12}=[1+i\sqrt{3}]^{12}=4096$$

Generalizing this approach one has that [a+ib]^n=[b+ia]^n when-

$$[a^2+b^2]^{n/2}\exp[in\arctan(b/a)]=[a^2+b^2]^{n/2}\exp[in\arctan(a/b)]$$

This means that-

$$n[\arctan(b/a)-\arctan(a/b)]=2\pi k \quad where \quad k=\pm1,\pm2,\pm3,etc$$

Furthermore using the identity-

$$\arctan(c) + \arctan(1/c) = \pi/2$$

leads to-

$$\arctan(\frac{b}{a}) = \frac{\pi}{4}\left[1 - \frac{4k}{n}\right]$$

This result can lead to multiple values of integer n for which the powers of the two complex numbers under consideration are equal. The requirement is seen to be that arctan(b/a) has the form π/integer. Thus, for example, b/a=sqrt(2)-1 produces arctan(b/a)=π/8 so that n/k=8 or n=8 when k=1. Therefore-

$$[(\sqrt{2}-1)+i]^8 = [1+i(\sqrt{2}-1)]^8 = \textbf{-1.88398…}$$

Also we note that arctan(b/a)=π/12 when b/a=[2-sqrt(3)]. Thus n=6k and-

$$[(2-\sqrt{3})+i]^6 = [1+i(2-\sqrt{3})]^6 = i[1.23122....]$$

See if you can derive the identity-

$$\left\{ [2-\sqrt{3}\,] + i[2\sqrt{2-\sqrt{3}}-1] \right\}^{24} = \left\{ [2\sqrt{2-\sqrt{3}} + i[2-\sqrt{3}] \right\}^{24}$$
$$= \text{-}0.2305755522... \cdot 10^{-13} \text{ - i } 0.1312798459... \cdot 10^{-20}$$

If one is willing to relax the restriction that n be an integer then many other solutions become possible. Indeed, recasting the above arctan criterion, one finds-

$$n = \frac{\mp k}{(\frac{1}{4} - \frac{\arctan(b/a)}{\pi})}$$

where n will now not necessarily be an integer. For a=1 and b=2, one has for k=-1 that-

$$[1+i2]^{9.764062902} = [2+i]^{9.764062902} = \text{ - } 476.2021596 \text{ - } 2540.353313\,\text{i}$$

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