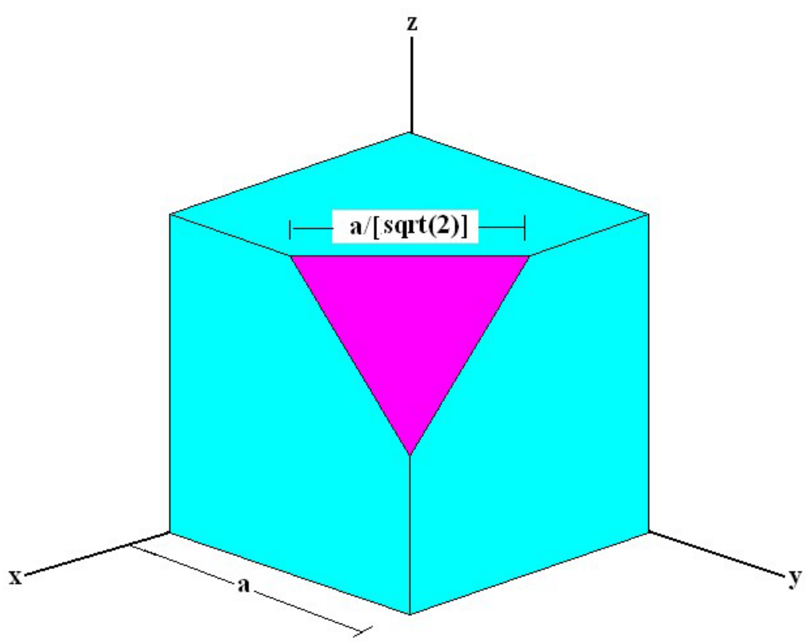
DÜRER'S CUBE

The picture shown at the beginning of our MATHFUNC mathematics page is the famous 1514 engraving Meloncholia by the German artist Albrecht Dürer(1471-1528). In addition to showing a 4x4 magic square with the date 1514 along the bottom line of the square, the picture also shows a peculiar solid which we will refer to as Dürer's Cube. It is also called a Rhombohedron or Dürer's Solid in the literature. Let us look at some of its properties.

Its basic structure consists of a solids cube of side-length 'a' out of which are cut two tetrahedrons in the form of a pyramid of triangular base with sides of length a/[sqrt(2)] and a slant height of a/2. One can construct such a 3D solid to yield the result shown-

DÜRER'S CUBE OF 1514



We have obtained this figure using the MAPLE plotting program-

```
A:=polygonplot3d({[[0,0,1],[0,1,1],[1/2,1,1],[1,1/2,1],[1,0,1]],[[0,1,1],[0,1,0],[1,1,0],[1,1,1/2],[1/2,1,1]],[[1,1,0],[1,0,0],[1,0,1],[1,1/2,1],[1,1,1/2]],[[1,0,1],[1,0,0],[1/2,0,0],[0,0,1/2],[0,0,1]],[[0,0,1],[0,1,1],[0,1,0],[0,1/2,0],[0,0,1/2]],[[1,0,0],[1,1,0],[0,1,0],[0,1/2,0],[1/2,0,0]]},axes=none,color=cyan,style=PATCH):

B:=polygonplot3d([[1,1/2,1],[1/2,1,1],[1,1,1/2]],color=magenta,style=patch):

C:=polygonplot3d([[1/2,0,0],[0,1/2,0],[0,0,1/2]],color=magenta,style=patch):

display(A,B,C,title=`DURER'S CUBE OF 1514`);
```

Not shown in this figure, but present in the actual calculation, is the back tetrahedron cut. One sees from the figure that the Dürer Cube has six faces in the form of identical but irregular pentagons plus two equilateral triangle faces. There are 18 edges and a total of 12 vertexes. That is-

Vertexes=V=18
Faces=F=8
Edges=E=18

which agrees with the Euler Geometric Formula V+F-E=2.

The volume of the Dürer Cube is –

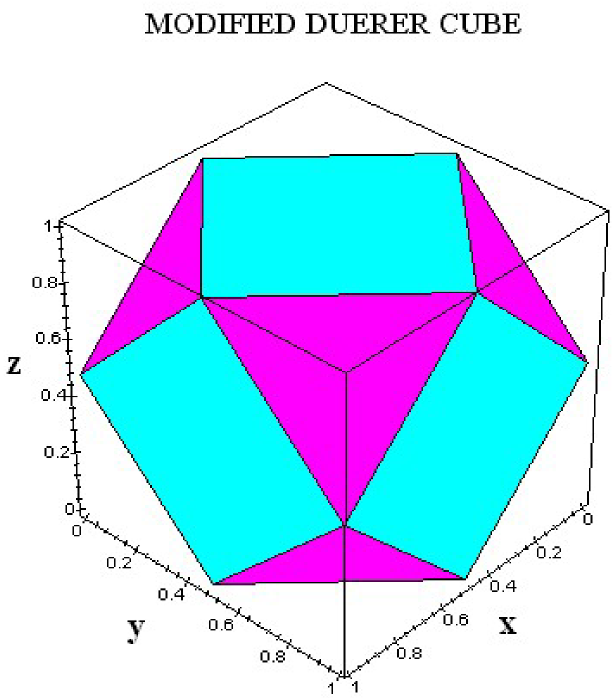
$$Volume = a^3 - 2[\frac{1}{8}(\frac{1}{3})(\frac{1}{2})]a^3 = [\frac{23}{24}]a^3$$

where again 'a' represents the length of the original uncut cube. The surface area is-

$$Area = 6(1-\frac{1}{8})a^2 + 2(\frac{\sqrt{3}}{8})a^2 = \frac{(21+\sqrt{3})}{4}a^2 = 5.6830127..a^2$$

and so is only slightly smaller than that for the uncut cube.

A variation on the Dürer Cube is one where one cuts off a tetrahedron at each of the 8 vertexes of the original cube. The result is a semi-regular polyhedron consisting of six square surfaces of sidelength a/sqrt(2)each plus eight equilateral triangles of the same sidelength. Looking at this modified Dürer Cube along the x=y=z line toward the origin yields the picture-



and is produced by the program-

```
A:=polygonplot3d({[[1,1/2,1],[1/2,1,1],[0,1/2,1],[1/2,0,1]],[[1,1/2,1],[1,1,1/2],[1,1/2,0],[1,0,1/2]],[[1,1,1/2],[1/2,1,0],[0,1,1/2],[1/2,1,1]]},axes=none,color=cyan,style=PATCH):

B:=polygonplot3d({[[1,1/2,1],[1/2,1,1],[1,1,1/2]],[[1,1,1/2],[1/2,1,0],[1,1/2,0]],[[0,1/2,1],[0,1,1/2],[1/2,1,1]],[[1/2,0,1],[1,1/2,1],[1,0,1/2]]},axes=none,color=magenta,style=PATCH):

display(A,B,title=`MODIFIED DUERER CUBE`);
```

The number of edges(E), faces(F), and vertexes(V) this figure has can be readily determined by looking at the one side of the figure shown. Counting the edges at the periphery as one half the number since these are shared with the part not visible, we find-

$$E = 2[\frac{1}{2}(6) + 6 + 3] = 24 \quad , \quad F = 2[7] = 14 \quad and \quad V = 2[\frac{1}{2}(6) + 3] = 12$$

Again the Euler Formula V+F-E=12+14-24=2 is satisfied. The volume is that of the unit cube minus eight times the volume of one of the cut-off tetrahedrons and the total surface area consists of eight equilateral triangles of area sqrt(3)/8 each plus six squares of area 1/2 each. That is-

$$Volume = 1 - 8[\frac{1}{48}] = \frac{5}{6} \quad , \quad Surface\ Area = \sqrt{3} + 3$$

These results are seen to be actually simpler than for the original Dürer Cube. One can attribute this to the greater symmetry possessed by the modified version. This modified version is recognized as one of the Archimedean semi-regular polyhedra and is often referred to as a cuboctahedron (see http://isotropic.org/polyhedra/) Note that this solid appears to be approaching a sphere. This will indeed be the case as one introduces further symmetric partitioning. You may recall that a typical soccer ball consist of a surface composed of 12 regular pentagons and 20 regular hexagons and very closely approximates a sphere. A discussion on the mathematics of soccer balls can be found at http://www.hoist-point.com/soccerball.htm .

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