## THE EULER-MASCHERONI CONSTANT AND THE HARMONIC SERIES

During our discussion on Bessel functions of the second kind we encountered the EulerMascheroni constant defined as-

$$
\gamma=\lim N \rightarrow \infty\left[\sum_{n=1}^{N}(1 / n)-\ln (N)\right]=0.5772156649 . .
$$

This important constant represents the difference between the harmonic series which is known to be divergent and the logarithm of infinity. Geometrically one has that the area between the curve $\ln (\mathrm{n})$ and the staircase function $1 /(\mathrm{n}+1)$ from $\mathrm{n}=1$ to infinity just equals 1 -gamma $=0.42278$.. .
Examining the function-

$$
G(N)=\sum_{n=1}^{N} 1 / n-\ln (N)
$$

for integer values starting with $\mathrm{N}=1$, one obtains a jagged curve which slowly approaches gamma for large N , but does so very slowly. Even at N of one million one has $\mathrm{G}(1,000,000)=$ 0.577216164 which is accurate to only five places. The question therefore arises - Is there a better way to find the value of the Euler-Mascheroni constant? The answer is yes by making use of the digamma function. In mathematical handbooks(such as that by Jan Tuma) one finds that the harmonic series up to the Nth term sums to-

$$
\sum_{n=1}^{N}(1 / n)=\sum_{n=1}^{\infty}(1 / n)-\sum_{n=1}^{\infty} 1 /(N+n)=\gamma+\Psi(N)
$$

where $\mathrm{psi}(\mathrm{N})$ is the digamma function defined as -

$$
\psi(N)=[\ln (\Gamma(N+1))]^{\prime}=\Gamma(N+1)^{\prime} / \Gamma(N+1)
$$

Now using the integral de finition of the gamma function-

$$
\Gamma(N+1)=N!=\int_{0}^{\infty} t^{N} \exp (-t) d t
$$

and differentiating this with respect to N , we have-

$$
\sum_{n=1}^{N}\left(\frac{1}{n}\right)=\gamma+(1 / N!) \int_{0}^{\infty} t^{N} \ln (t) \exp (-t) d t
$$

That is, the digamma function $\mathrm{psi}(\mathrm{N})$ is represented by a single integral which can be readily evaluated for a given N to find the Euler-Mascheroni constant. Let us demonstrate. Using a numerical evaluation of the integral when $\mathrm{N}=5$, one finds-

$$
\gamma=(1+1 / 2+1 / 3+1 / 4+1 / 5)-\int_{0}^{\infty} t^{5} \ln (t) \exp (-t) d t / 120
$$

which has the value $\mathbf{1 3 7 / 6 0 - 1 . 7 0 6 1 1 7 6 6 8 4 2 9 2 4}=\mathbf{0 . 5 7 7 2 1 5 6 6 4 9 0 4 0 9 5}$. The result gives the exact first eleven decimal places of gamma with dep artures after that due to the limitations of the Mathcad program used in the integral evaluation We also have, at $\mathrm{N}=1$, that-

$$
\int_{0}^{\infty} t \ln (t) \exp (-t) d t=\psi(1)=1-\gamma=0.422784335 \ldots
$$

In addition, one finds that the sum of the reciprocal of the first 100 integers will be -

$$
\sum_{n=1}^{100}\left(\frac{1}{n}\right)=\gamma+\int_{0}^{\infty} t^{100} \ln (t) \exp (-t) d t=5.18737751763962 \ldots
$$

Its really quite surprising that there is not a simpler formula for determining the value of the sum of the reciprocal of the first N integers, when straightforward summations for the sum of the first N terms of the integers with positive powers exist. For example, we know that -

$$
\sum_{n=1}^{N} n^{4}=\left[6 N^{5}+15 N^{4}+10 N^{3}-N\right] / 30
$$

so that the sum of the fourth power of the first ten integers is simply twenty five thousand three hundred and thirty three.

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