

## PYTHAGOREAN TRIPLETS AND AN EXTENDED PYTHAGOREAN THEOREM

A Pythagorean triplet  $[a,b,c]$  represents the lengths of the sides of a right triangle where  $a$ ,  $b$ , and  $c$  are integers. Simple examples are  $[3,4,5]$  and  $[5,12,13]$ . One can show that these integer triplets can be generated by-

$$a = 2\Delta n, \quad b = n^2 - \Delta^2, \quad \text{and} \quad c = n^2 + \Delta^2$$

,where  $n$  and  $\Delta$  are any positive integers and  $n > \Delta$ . Thus for  $n=21$  and  $\Delta=12$  we have as a possible triplet the group  $[504,297,585]$ . The base triplet corresponding to this result is found by dividing by the greatest common denominator of the group which is here 9. Thus the base triplet is  $[56,33,65]$  so that  $n=7$  and  $\Delta=4$ .

In thinking about the Pythagorean Theorem many mathematicians have looked at extended versions such as –

$$a^n + b^n = c^n$$

,when  $n$  is greater than two. It was Fermat who in 1635 first conjectured that such equations can not have integer solutions whenever  $n \geq 3$ . Many attempts were made to disprove this conjecture (also known as Fermat's Last Theorem) throughout the years. It was not until recently (Wiles,1995) that Fermat was proven right.

What we wish to point out here is that if one relaxes the requirement that all terms have the same power  $n$  in the exponents, multiple integer solution triplets become possible. That is, there should be an infinite number of solutions possible which satisfy the modified Fermat-Pythagorean Formula

$$a^n + b^m = c^k$$

, where  $n$ ,  $m$ , and  $k$  are positive integers exponents not all three equal. Also  $a$ ,  $b$ , and  $c$  remain integers as in the original Pythagorean Theorem. We want here to show that an infinite number of such triplets exist.

We start with the very simple examples –

$$2^4 + 3^2 = 5^2, \quad 2^2 + 2^5 = 6^2, \quad 3^1 + 5^3 = 2^7 \quad \text{and} \quad 3^3 + 6^3 = 3^5$$

which will be designated as the triplets -

$$[2^4, 3^2, 5^2] \quad , \quad [2^2, 2^5, 6^2] \quad , \quad [3^1, 5^3, 2^7] \quad \text{and} \quad [3^3, 6^3, 3^5]$$

These elementary triples are easiest to find by use of a power table such as the following-

N	n <sup>2</sup>	n <sup>3</sup>	n <sup>4</sup>	n <sup>5</sup>	n <sup>6</sup>	n <sup>7</sup>
1	1	1	1	1	1	1
2	4	8	16	32	64	128
3	9	27	81	243	729	2187
4	16	64	256	1024	4096	16384
5	25	125	625	3125	15625	78125
6	36	216	1296	7776	46656	279936
7	49	343	2401	16807	117649	823543

Note that the first triplet is equivalent to the standard 3-4-5 triangle of Pythagoras while the last is not a base triplet since it is divisible by 27=3<sup>3</sup> to yield the true base triplet-

$$[1^1, 2^3, 3^2]$$

Base triplets result after dividing out all the common factors in a given triplet.

One can also reverse things and multiply a base triplet by certain numbers N to produce a larger triplet. This multiplier must be chosen carefully. The simplest way to find such triplets is to multiply the base triplet by increasing Ns one unit at a time until a new triplet is found. Let us demonstrate the procedure for the base triplet [2<sup>4</sup>,3<sup>2</sup>,5<sup>2</sup>]. We find on multiplying by 2<sup>2</sup>, 2<sup>4</sup>, 2<sup>6</sup>,etc the triplets-

$$[2^6,6^2,10^2] \quad , \quad [2^8,12^2,20^2] \quad \text{and} \quad [2^{10},24^2,40^2]$$

These generalize to the infinite set of triplets-

$$[2^{4+2n}, (3 \cdot 2^n)^2, (5 \cdot 2^n)^2]$$

From these one has, for instance, that-

$$4194304=(2560)^2-(1536)^2$$

at n=9. We may also multiply this base triplet by 3<sup>2</sup>=9, 3<sup>4</sup>=81, etc to get the triplets-

$$12^2 + 3^4 = 15^2 \quad \text{and} \quad 6^4 + 3^6 = 45^2$$

whose generalization reads-

$$(12 \cdot 3^n)^2 + 3^{4+2n} = (15 \cdot 3^n)^2$$

It is obvious that one can generate still more triplets by multiplying the base triplet by numbers such as 25, 49, etc

We can also employ the other base triplet  $[1^1, 2^3, 3^2]$  to generate an infinite number of higher non-base triplets. Some of these are-

$$4^1 + 2^5 = 6^2, 4^2 + 2^7 = 12^2, \text{ and } 4^3 + 2^9 = 24^2$$

So that its general form reads-

$$4^n + 2^{3+2n} = (3 \cdot 2^n)^2$$

At  $n=6$ , we have the triplet-

$$[4^6, 2^{15}, 192^2]$$

Rearranging, one can write this last result as –

$$2^{15} - 1 = 192^2 - 4^6 - 1 = 32767 = 7 \cdot 31 \cdot 151$$

which happens to be one of the Mersenne Numbers which is not a prime. Using  $n=8$  we get the equality-

$$2^{17} - 1 = (768)^2 - 4^8 - 1 = 131071$$

which is a prime number.

The above infinite sets of Fermat-Pythagorean Triplets found based on just three base triplets does not exhaust the possibility of the existence of other triplets such as  $[3^5, 10^2, 7^3]$  being found using a larger power table. It does, however, clearly indicate that-

$$a^n + b^m = c^k$$

has an infinite number of solutions for integer  $a, b, c, n, m,$  and  $k$  provided that at least two of the three integer exponents differ from each other. The base triplets are determined from a power table and their large number off spring are found by multiplying the original triplet by appropriate values of  $N$  typically of the form  $N=(\text{integer})^n$ . The identity -

$$3^{17} + 7290^2 = 567^3$$

follows from one of the fundamental base triplets and is found by dividing the expression by the gcd of the three terms.

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