GENERATING PRIME NUMBERS WITH F[n]=n^2+(n+1)^2

One of the best known prime number generators is the Mersenne Formula N[n]=2^n -1. There have been some 47 values of n found for which the number N[n] will be prime. Many other formulas which one may use to generate primes exist. Among these one finds the polynomial prime number generator -

$$N[n] = n^2 + (n+1)^2 = 2n(n+1) + 1$$

This can also be written in the alternate form-

$$N[1] = 5 \ with \ N[n+1] = N[n] + 4(n+1)$$

Note that it produces an odd number for all positive integer values of n and many of these will be prime. Running the simple one line program-

for n from 1 to 200 do {n, 2*n*(n+1)+1, isprime(2*n*(n+1)+1)}od;

produces the following table for the first 60 primes of N[n].

n	N[n]	n	N[n]	n	N[n]
1	5	47	4513	115	26681
2	13	50	5101	122	30013
4	41	60	7321	130	34061
5	61	65	8581	135	36721
7	113	69	9661	137	37813
9	181	70	9941	139	38921
12	313	72	10513	144	41761
14	421	79	12641	149	44701
17	613	82	13613	154	47741
19	761	84	14281	157	49613
22	1013	85	14621	160	51521
24	1201	87	15313	162	52813
25	1301	90	16381	164	54121
29	1741	97	19013	172	59513
30	1861	99	19801	174	60901
32	2113	100	20201	185	68821
34	2381	102	21013	187	70313
35	2521	104	21841	189	71821
39	3121	109	23981	195	76441
42	3613	110	24421	199	79601

Note that these primes are much more closely spaced than the Mersenne primes and that they all end in the integer 1 or 3 with the exception of N[1]=5. A large 200 digit prime number generated by this formula for –
n :=
845621376839234046567490234382659497896347821572874395270173497375679824959593467309195343784540938173463856808123656079
is-
N[n]=1430151025934963750386424245296996875112195903891119629608626183507475919634614234127318528416215012952850781936572707911540354989749150667797717561281358608227724689027455777151923740880861795932522395211646953899785623599337238595994620641

Note that are many values of n for which N[n] will not be prime. First of all all numbers N[n] ending in 5 will be composite, while those ending in 1 or 3 may or may not be. They will require testing for primeness. One way to do this is to make use of Fermat's little theorem which states that a number is prime if-

$$[2^{N(n)-1}-1]/N(n)=Integer$$

Thus , for example, at n=112 we have N(112)=25313. In this instance the Fermat Theorem yields-

[2^111-1]/112=370878347038201973466464023515721/16

This being a fraction makes 25313 a composite number. As with all known prime tests, this procedure becomes impractical when N becomes large because of the magnitude of the number generated by the Fermat quotient. Alternatively one can carry out a brute force test using the simple ratio-

$$R[n,m]=\frac{N[n]}{2m+1}\quad m=1,\,2,\,3,...$$

If one finds no values of m less than about 0.5[sqrt(N(n))-1] for which R[n,m] is an integer, then N[n] is prime. Testing things for N[112]=25313 we find R[112, 8]=1489. Hence 25313=17x1489 is composite. Again for very large values of n such an evaluation becomes cumbersome. It is, however, the only known way to factor general large composite numbers.

To demonstrate an accelerated version of this last factoring technique, consider the number –

N[11523]=2661125

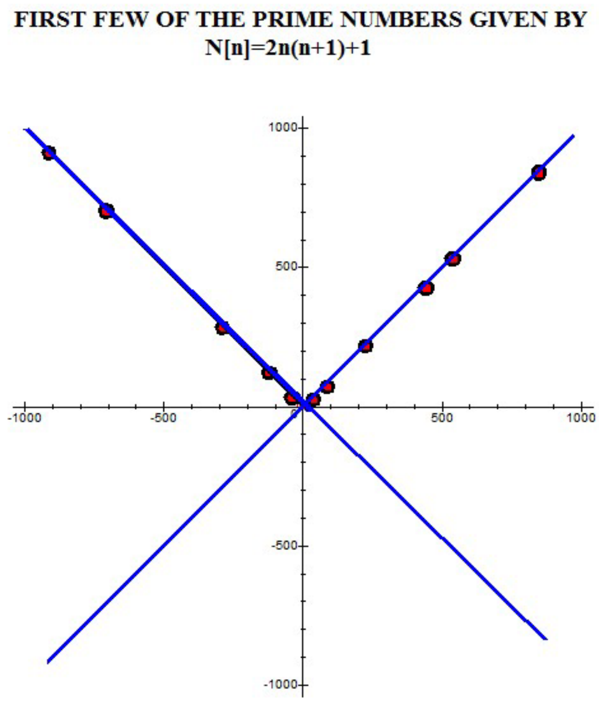
It is obviously composite because of the five ending. To factor it, we first perform the division N[n]/(2m+1) for m=1 to 100 and note integer quotients when m=2, 12,

30, and 62 in the range of m considered. Next we take the product 2(30)+1 and 2(62)+1 of the two largest ms found and divide this into 2661125. This leaves us with the smaller number 349. Then we look at the quotient 349/(2m+1) for m=1 to m=18. Carrying out the evaluations, one finds no further integer values. Hence we conclude that the composite number-

$$N[11523]= 2661125=61*125*349=5*5*5*61*349$$

expressed as the product of prime numbers. This procedure can be automated as a simple program in MAPLE and is found to work well for numbers as large as one hundred digits. The number of required operations again will become prohibitive for most computers when dealing with billion and trillion digit numbers N(n).

Finally, going back to the above table of primes, we check where these fall in our cross diagonal pattern as developed in some of our previous notes. A point-plot of the first few of these primes leads to the following pattern-



As seen, they all lie along the 45 degree diagonal in the first and third quadrant. The Mersenne primes 3,7, 31, 127, 8191, etc have a much sparser distribution and all lie along the diagonal in the 4^{th} quadrant and end in the digits 1 or 7 except for the first M[2]=2^(2-1)-1=3.

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