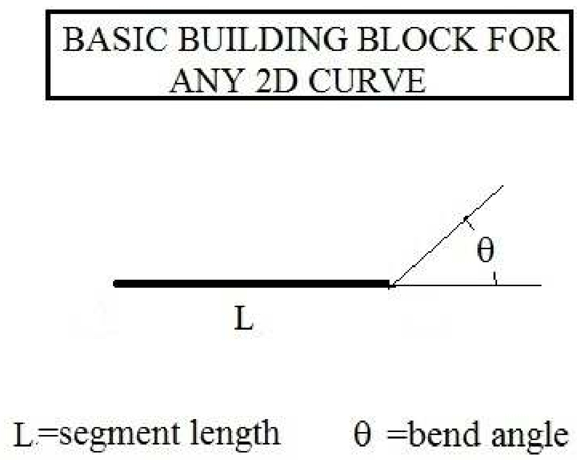
FORMULAING GENETIC CODES FOR ANY 2D CURVE

In a recent note(Feb. 2012) we showed how one can produce certain 2D figures of unit edge length from simple codes in a manner not unlike what occurs in DNA reproduction. We want here to extend these discussions to more complicated 2D structures involving variable length sides and arbitrary but specified connection angles. Our basic building block will here be the following line segment L and its specified connection angle θ -



Our designation for this building block of a 2D code will be [L,θ] with positive angle being a counterclockwise measure and a negative angle indicating a clockwise orientation of the next building block. A very simple figure, the square, is produced by concatenating the building block [L,π/2] a total of four times as follows-

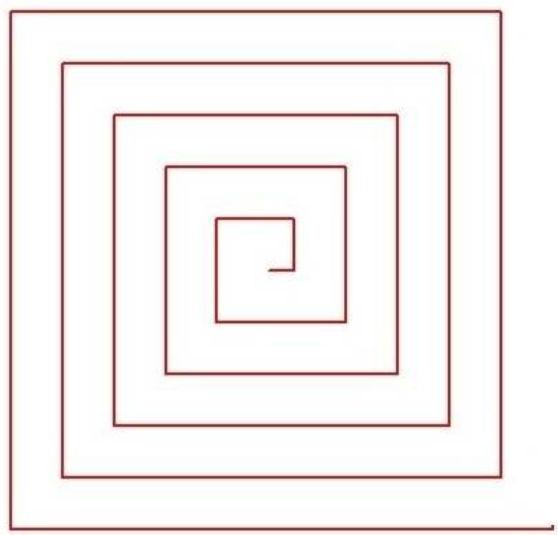
$$[L,\pi/2]\text{-}[L,\pi/2]\text{-}[L,\pi/2]\text{-}[L,\pi/2]$$

Note that this square has sides of length L and all four angles add up to 2π radians as expected for a closed figure. An interesting property of such genetic codes is that they apply for any orientation of the 2D figure and the location of the figure relative to the zero coordinate point is immaterial. That is, such codes are independent of the figure location and orientation. Let us next try a somewhat more difficult problem where the basic building block for the genetic code is [n, π/2] . Starting with n=1 we get the concatenation-

$$[1, \pi/2]\text{-}[2, \pi/2]\text{-}[3, \pi/2]\text{-}[4, \pi/2]\text{-}[5, \pi/2]\text{-} \ldots$$

What is this? Clearly the block length increases monotonically with n and the angle between neighboring line elements remains always at 90 degrees counter-clockwise. The 2D figure created by this code will thus become of infinite size as n goes to infinity. If we start our unit length element at x=y=0, the first corner occurs at [x,y]=[1,0], the second at [1,2], the third at [-2,2], and the fourth at [-2,-2]. There is no easy way to represent these points by a general locus formula in Cartesian coordinates, but the code clearly specifies where these corners are. Carrying out a listplot operation with our MAPLE math program , one finds the following interesting 2D figure-

SQUARE SPIRAL
(With Building Block $[n, \pi/2]$)



This figure has no particular name although square-spiral would not be inappropriate. Again the same code applies to this figure when rotated about any angle and its shape will not be changed when using a different starting point than [0,0] . To magnify the figure one just needs to choose a larger L. The figure is reminiscent of some designs found on ancient Greek temples and similar to inductor shapes in microcircuits. It is also reminiscent of certain crystal structures found near lattice dislocations. I recall meeting J.M.Burgers (of Burgers equation fame) several decades ago while he was giving us a seminar here at UF. After the seminar I drove him over to St.Augustine to look at the old Spanish Fort. During dinner there he was telling me all about dislocations and some of the studies he had done along these lines(Burgers Vector in crystallography) years earlier.

Consider next an oblique triangle with sides a=2, b=3, and c=4. What is the genetic code for this figure? Clearly it requires three building blocks. The first element can be taken as $[2, \theta_1]$, the second as $[3, \theta_2]$, and the third is$[4, \theta_3]$. We also have from the law of cosines that θ_1=arcos(1/4), θ_2 =arcos(-11/16), and θ_3=arcos(-7/8). These angles add up to 2π radians as expected. The code for this closed triangle is thus-

[2, arcos(1/4]-[3, arcos(-11/16)]-[4, arcos(-7/8)]

Its also a simple matter to find the code for any triangle abc. A bit of manipulation shows it to be given by-

$[a, \arccos((c^2-a^2-b^2)/(2ab))]-[b, \arccos((b^2-a^2-c^2)/(2ac))]-[c, \arccos((a^2-b^2-c^2)/(2bc))]$

The genetic code for the swastika consists of five basic elements and reads-

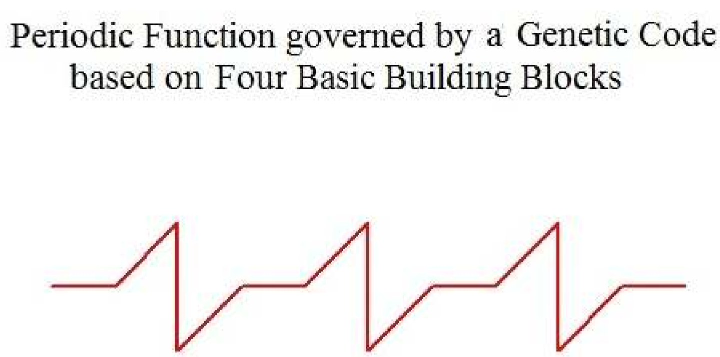
$$[1,-\pi/2]-[2,\pi/2]-[1,\pi/2]-[3,\pi/2]-[2,-\pi/2]$$

This code must be repeated four times to generate the complete figure.

Look next at the four basic building block code-

$$[1, \pi/4]-[\mathrm{sqrt}(2),-3\pi/4]-[2, 3\pi/4]-[\mathrm{sqrt}(2), -\pi/4]$$

If you repeat this code several times one obtains the open ended curve-



Just like in a real genetic code this pattern can be extended indefinitely all the way from minus infinity to plus infinity. Also it is superior to the use of Fourier series for periodic functions especially when the function has discontinuities. Consider the following discontinuous periodic function

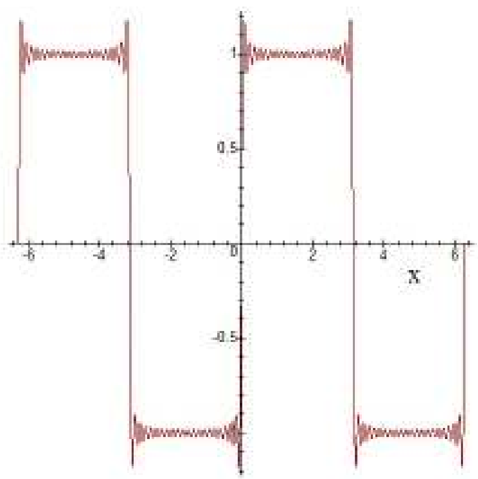
$F(x)=+1$ when $2n<x<2n+1$ and $F(x)=-1$ when $2n+1<x<2n+2$

Its genetic code reads $[1, -\pi/2]-[2, \pi/2]-[1, \pi/2]-[2, -\pi/2]$ and will reproduce this function indefinitely. A Fourier sine series for this same function reads-

$$F[x]=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{(2n+1)}$$

It is is a very slowly convergent series for F(x) and produces the Gibbs overshoot phenomenon at integer values of x. Summing out to n=20 produces the result-

FOURIER SINE SERIES FOR THE PERIODIC FUNCTION
F(x)=+1 for 0<x<1 and F(x)= -1 for 1<x<2



It clearly departs from the true value of F(x) near integer values of x. This departure becomes quite small as n approaches infinity but an overshoot remains. The genetic function has no problems in this regard since it easily handles function discontinuities.

Continuous functions may also be created by the generic code mechanism using an appropriate limiting procedure . Consider the building blocks of length L= 2 sin(π/N) representing an N sided regular polygon where the distance to the polygon center is one.. In this case the basic building block reads-

$$[2\sin(\pi/N), 2\pi/N]$$

If we hook N+1 of these blocks together a closed regular polygon will result and if we let N approach infinity a continuous circle of radius 1 centered on the origin will result. It is this type of limiting procedure which Archimedes and Ludolph van Ceulen used to calculate π prior to the invention of calculus. We have carried out such a calculation for N=20 using the MAPLE command-

```
with(plots):
listplot([seq([1,2*Pi*n/20],n=1..21)],coords=polar,color=black,axes=
none,thickness=2,numpoints=2000,scaling=constrained);
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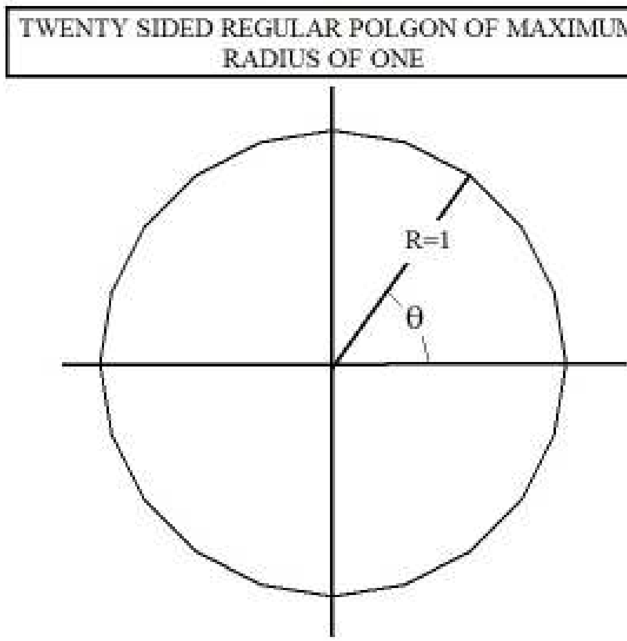
The resultant polygon looks as follows-



Genetic Building Block is [2sin(π/N), 2π/N]

Note that we have used a polar coordinate system for the plot. It is more convenient than a Cartesian representation for such an axisymmetric situation. If one lets N gets larger and larger the polygon yields a continuous circle. Its circumference is 2π and this should coincide with the polygon total side-length of $N(2\sin(\pi/N)$ as $N\to\infty$. That is-

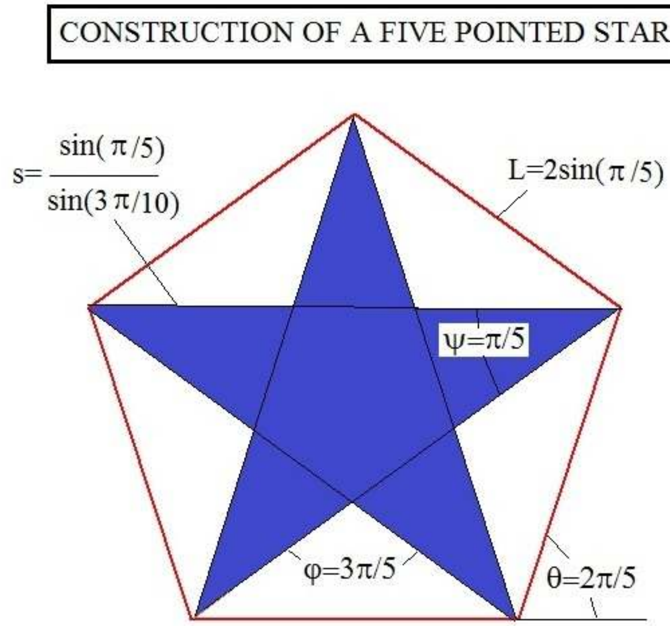
$$2\pi = \lim_{x\to\infty}\{2N(\sin(\pi/N)\}$$

or the equivalent that $\sin(\varepsilon)/\varepsilon=1$ as $\varepsilon=\pi/N$ goes to zero, a well known result from calculus.

Finally consider constructing a star from a regular polygon. For this purpose we start with a regular pentagon whose genetic building block is-

$$[L,\theta]=[2\sin(\pi/5), 2\pi/5]$$

when the distance from the pentagon center to each of its five outer corners is set to unity. Connecting every second corner by a straight line then produces the five pointed star shown-



The dimensions of the various parts of the star are as indicated. There are two fundamental blocks for the genetic code of this star. They read-

$$[s, -2\pi/5]\text{-}[s, 4\pi/5] \qquad \text{where} \qquad s= \sin(\pi /5)/\sin(3\pi/10)$$

They will reproduce the five pointed star shown in blue for any rotation angle. Note that-

$$5(-2\pi/5+4\pi/5)=2\pi$$

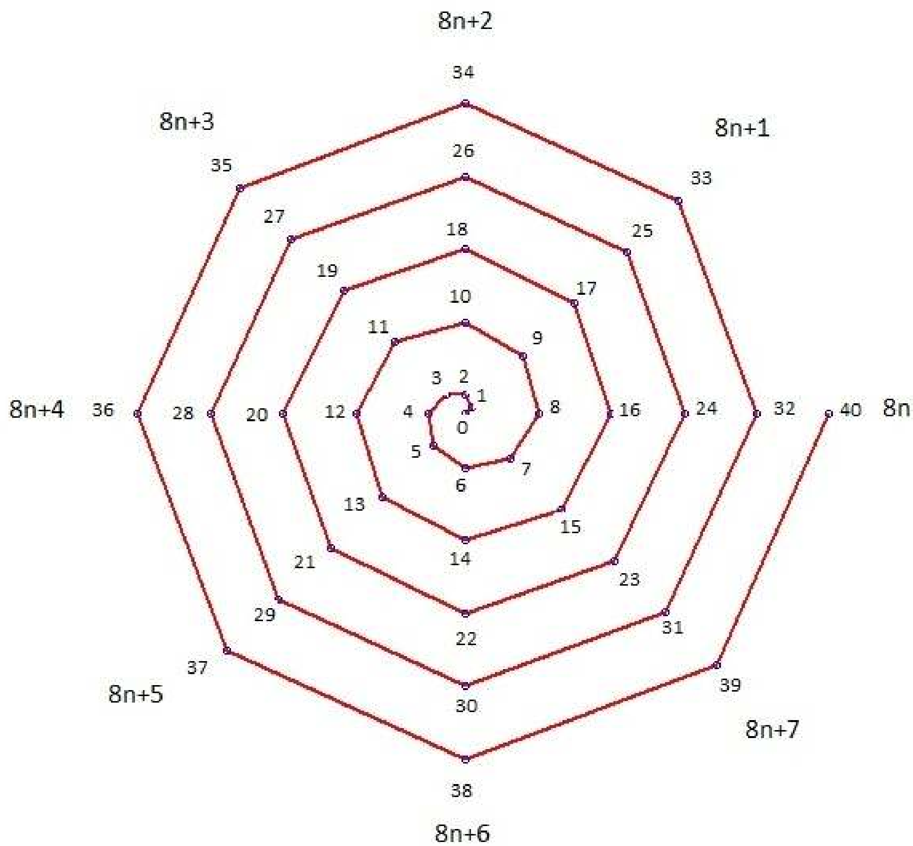
as expected.

Finally, one can also generate certain 2D curves by the building block [r,θ] where r represents the radial distance from the origin to a corner on the curve and θ the polar angle this line makes with respect to the x axis. A prime example of such a curve is the integer spiral discovered by us several years ago. There we have the genetic building block [r,θ]=[n, πn/4]. Writing out the sequence we get-

$$[0,0]\text{-}[1,\pi/4]\text{-}[2,\pi/2]\text{-}[3,3\pi/4]\text{-}[4,\pi]\text{-}[5,5\pi/4]\text{-} \;\ldots.$$

It yields the following plot-

INTEGER SPIRAL



Note the interesting grouping. Here all odd integers lie along the two diagonals and all the even integers along either the x or y axis.

September 2012