GENERATING ROTATIONALLY SYMMETRIC FIGURES FROM THE BASIC ITERATION $Z_{n+1} = \sum_{k=0}^{N} C_{2k} {Z_n}^{2k}$

In an earlier discussion on symmetry(see RIC'S TECH-BLOG) we noticed that an iteration of the type Z[n+1}=Z[n]^6 +C produces a figure with a six-fold rotational symmetry in that portion of the complex Z plane where the Z[∞] is bounded. We want here to extend this discussion by looking at the more general iteration-

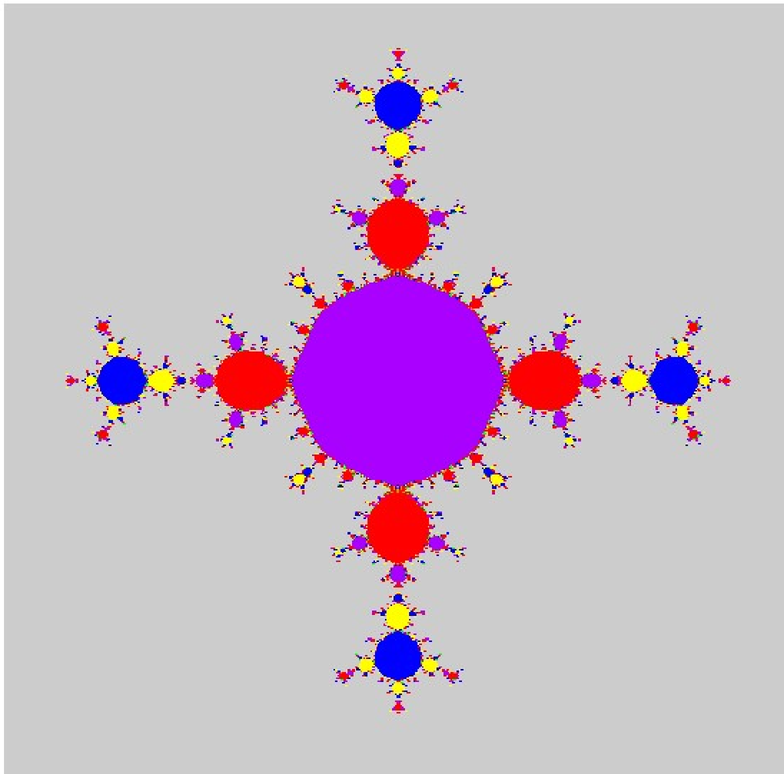
$$Z_{n+1} = C_0 + C_2 {Z_n}^2 + C_4 {Z_n}^4 + C_6 {Z_n}^6 + \ldots + C_{2N} {Z_n}^{2N}$$

involving the linear sum of all even powers of Z up to 2N. Here the Cs are given complex numbers and N is specified before hand. Our choice of using only the even powers of Z in this iteration is that we want to maintain at least a bilateral (ie two-fold) symmetry in the resultant figures. This means that the higher powers of Z should be divisible by Z^2. In working out this iteration we use a modification of the following MATLAB fractal program found at **http://eulero.ing.unibo.it/** . It reads-

```
col=20;
m=400;
cx=0;
cy=-0;
l=1.5;
x=linspace(cx-l,cx+l,m);
y=linspace(cy-l,cy+l,m);
[X,Y]=meshgrid(x,y);
c=-1.14+0.0i;   ← this is replaced by C0 from above
Z=X+i*Y+eps;
for k=1:col;
Z=Z.^2+c;   ← modify this term to the form of the above expansion
W=exp(-abs(Z));
end
colormap prism;
pcolor(W);
shading flat;
axis('square','equal','off');
```

Let us begin with an evaluation of W=exp(-abs(Z)) for the special case of C_0=-1.14, C_2=0, and C_4=1 with N=2. One finds the following figure-

ITERATION $Z[n+1]=Z[n]^4$ -1.14

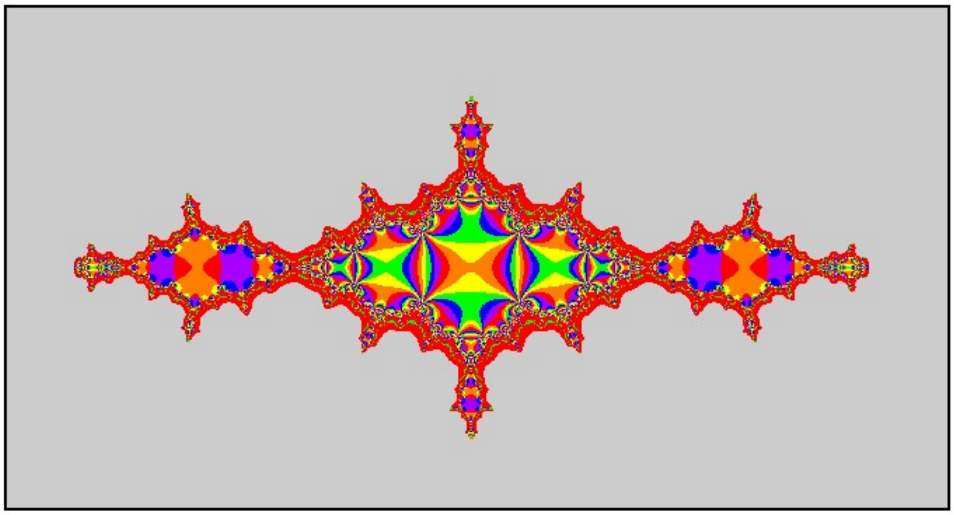


The light grey portion of the figure shows the region within the Z plane where the iteration blows up. The color portions indicate regions in the Z plane where the iteration is bounded although it will generally not converge to a unique single value but rather assume several different distinct values for the sequence Z[n+1], Z[n] , Z[n-1], and Z[n-2] as n gets large. Let us demonstrate by picking three different starting points for Z[0] but keeping C_0=-1.14. The results are-

Z[0]	Z[198]	Z[199]	Z[200]	Z[201]
1.2	0.072224	-1.13997	0.54879	-1. 04929
0.4 I	0.54879	-1. 04929	0.072224	-1.13997
0.3(1+I)	0.072224	-1.13997	0.54879	-1. 04929

Clearly the iterations are bounded for the value of C used and will jump from one convergent value to another on the next iteration. We have found this type of bounded but not uniquely convergent behavior in an earlier iteration study found on the present MATHFUNC page (see PATHS TO CONVERGENCE FOR THE MADELBROT SET). Regardless of the initial value Z[0] the iterations always end up in this case with the four values given in the table.. The figure has a four-fold rotational symmetry which follows from the fact that replacing $Z=r \exp I\theta$ by $Z=r \exp I(\theta+\pi/4)$ will only change the sign of Z^4 and hence makes no difference when plotting `W=exp(-abs(Z))`as stipulated in the program. The different colors in the figure indicate different non-zero values of W. Notice if one were to stop the above expansion after N=1, the iteration would reduce to the classic Mandelbrot form which in the present way of plotting W yields (for the special case $C_0=-1.22$ and $C_1=1$) the plot-

ITERATION $Z[n+1]=-1.22+Z[n]^2$

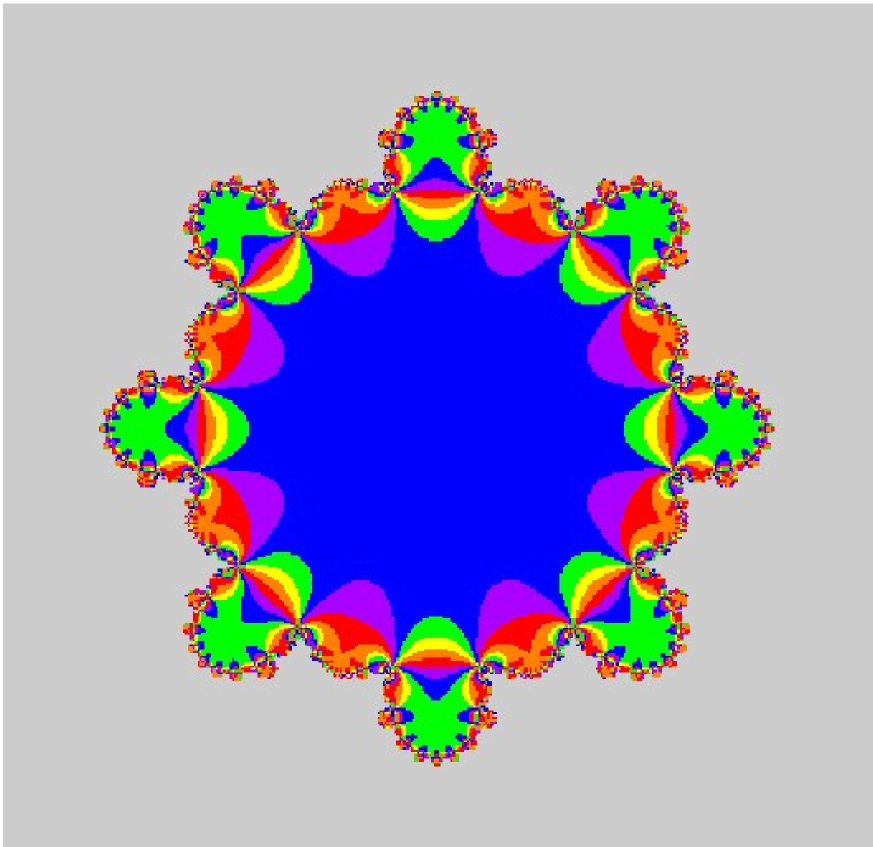


plot shows W=exp(-abs(Z[inf]))

The complexity of this bilaterally symmetric color pattern is quite astounding considering the simple iteration which produces it.

Take next the iteration-

Z[n+1]=Z[n]^8-0.78

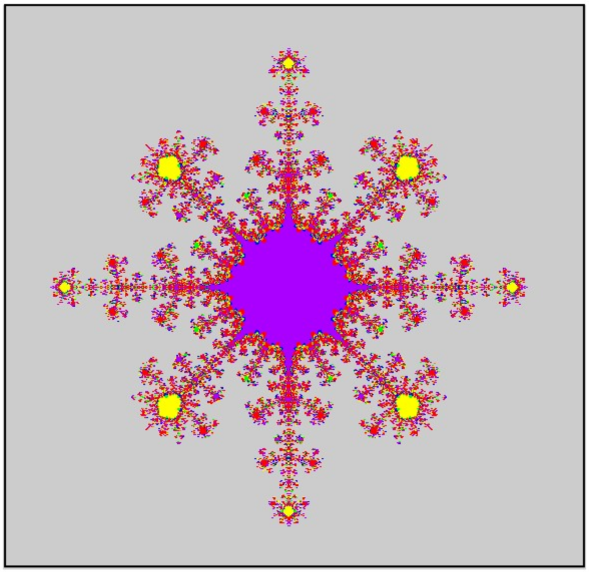


It produces an eight-fold symmetric figure in the Z plane where the iteration yields bounded value. The reason for this symmetry is that the lowest non-vanishing power in the iteration is eight. Note that if the pattern is taken to represent a decorated pizza, there would be eight identical slices each $\pi/4$ radians of rotation wide.

Consider next an iteration involving the 0th, 4^{th} , and 8^{th} power of Z, namely,-

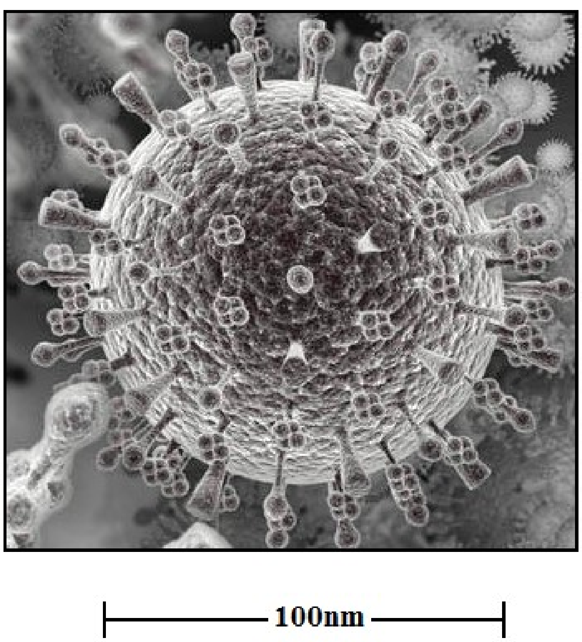
$$Z_{n+1} = -0.755 - Z_n^{\ 4} + Z_n^{\ 8}$$

That is, C_0=-0.755, C_2=-1, C_4=+1, and N=4. This time one expects to find a four fold rotationally symmetric figure. This is indeed the case as shown-



The intricacy of this figure is again quite amazing considering the simple formula used in its construction. It calls to mind the complicated structure of viruses and bacteria whose DNA blue print must be very simple considering their size. It also suggests that modified versions of the type of iteration we are considering here could find applications in generating interesting 3D structures. For example it might be possible to generate the intricate structure of something like the bird flu virus shown-

Avian Flu Virus



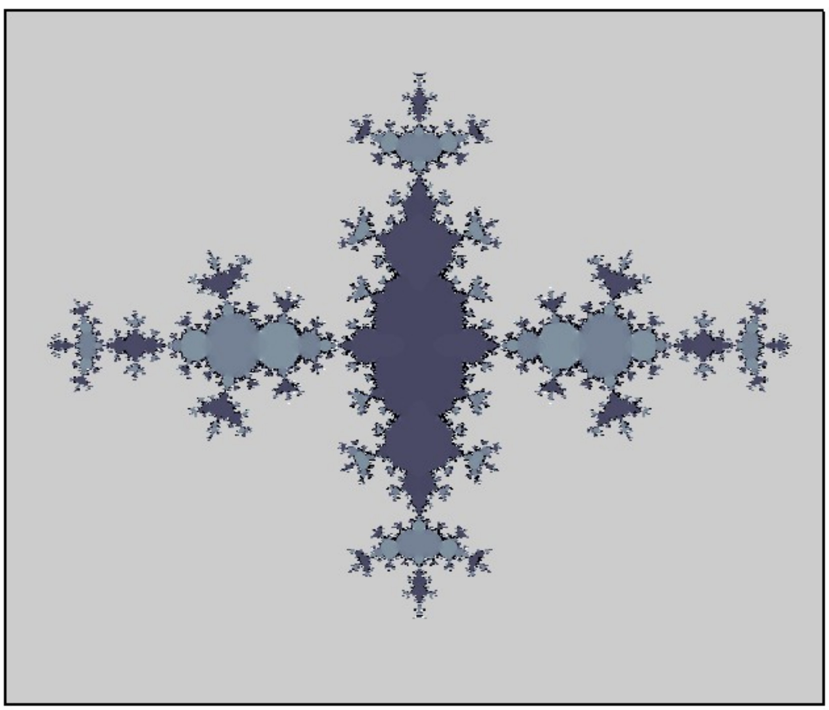
by some type of simple 3D iteration

As another choice , we examined the special N=2 case-

$$Z[n+1] = -0.871 - Z[n]^2 + Z[n]^4$$

The resultant pattern looks like this-

FUNCTION EXP(-ABS(Z[inf])) IN GRAY SCALE
FOR ITERATION $Z[n+1]= -0.871-Z[n]^2 + Z[n]^4$

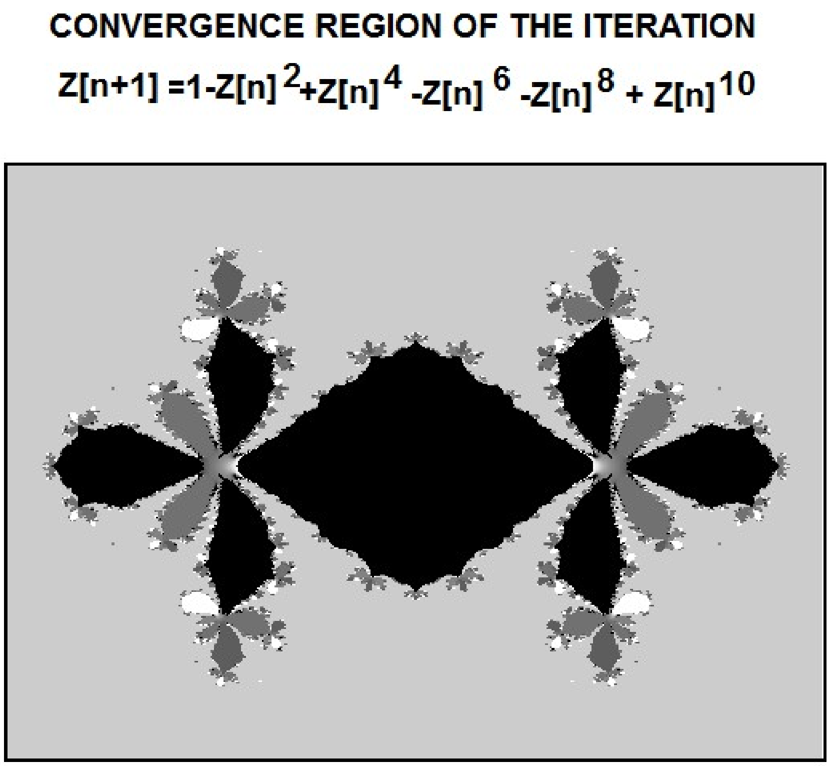


Here we have used a gray scale with the darker gray indicating boundedness (W is finite) and the lightest gray indicating divergence (W=0).

Finally we examined the more complicated N=5 iteration-

$$Z_{n+1} = 1 - Z_n^{\ 2} + Z_n^{\ 4} - Z_n^{\ 6} - Z_n^{\ 8} + Z_n^{\ 10}$$

Plotting the term W=exp(-abs(Z[inf])) one obtains the unique two-fold symmetric figure-



The black and dark grays indicate convergence and the light gray (W=0) divergence. I have made an acrylic interpretation of this last iteration. Here is what it looks like-

The picture colors and white background show up much better in the original drawing.

There are an infinite number of other bilateral patterns which can be created by the above iteration involving different combinations of powers of Z and specified values of the complex constants C=a+ib. Some of these turn out to be interesting others not.

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