THE LEGENDRE POLYNOMIALS AND THEIR PROPERTIES

The gravitational potential ψ at a point A at distance r from a point mass located at B can be represented by the solution of the Laplace equation in spherical coordinates. In its simplest form one has-

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\psi}{\partial r} \quad with \quad solution \quad \psi(A) = \frac{Const.}{r}$$

If one now thinks of obtaining the potential of a distributed mass, the solution becomes-

$$\psi(A) = const.\iiint\frac{dVol}{r} = \iiint\frac{dVol}{R_0\sqrt{1+\rho^2-2\rho\cos(\theta)}}$$

Where R_0 is the distance from the center of mass at C to the external point A , $\rho=R/R_0<1$ is the ratio of distance BC to AC, and θ is the angle between lines AC and BC. As first done by Legendre, one can expand the radical in the denominator of this integral to get-

$$\frac{1}{\sqrt{1+\rho^2-2\rho\cos(\theta)}} = 1+\rho[\cos(\theta)]+\rho^2[\frac{3\cos(\theta)^2-1}{2}]+\rho^3[\frac{5\cos(\theta)^3-3\cos(\theta)}{2}]+$$

$$\rho^4[\frac{35\cos(\theta)^4-30\cos(\theta)^2+3}{8}]+\rho^5[\frac{63\cos(\theta)^5-70\cos(\theta)^3+15\cos(\theta)}{8}]+.....$$

Substituting this expansion into the above integral leads to the well known multipole expansion for the potential of a gravitational mass as viewed from a point A outside of this mass. For us the important point here are the polynomials appearing in the square brackets of the above expression. These are the Legendre Polynomials usually expressed in terms of the variable x=cos(θ). They read-

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$$

$$P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$$

Since θ in the potential problem ranges from –π to +π the range of x of interest will be

-1<x<+1. Note that P_{2n} are even functions while P_{2n+1} are odd. One can develop a generating function for these Legendre polynomials starting with-

$$\frac{1}{\sqrt{1+\rho^2-2\rho x}} = \sum_{n=0}^{\infty} \rho^n P_n(x)$$

Differentiating once with respect to ρ we have-

$$\frac{(x-\rho)}{[1+\rho^2-2\rho x]^{3/2}} = \frac{(x-\rho)}{[1+\rho^2-2\rho x]} \sum_{n=0}^{\infty} \rho^n P_n(x) = \sum_{n=0}^{\infty} n\rho^{n-1} P_n(x)$$

or the equivalent form-

$$\sum \rho^{n+1}\left[(n+2]P_{n+2}(x) + (n+1)P_n(x) - x(2n+3)P_{n+1}(x)\right] = 0$$

We thus have the generating formula for Legendre polynomials-

$$P_{n+2}(x) = \frac{[-(n+1)P_n(x) + x(2n+3)P_{n+1}}{(n+2)}$$

This formula is easy to program starting with P_0=1 and P_1=x.

One finds, for example, that-

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

A differential equation for Pn(x) can be found by examining the function-

$$F(n,x) = (1-x^2)\frac{dP_n(x)}{dx} = (1-x^2)\frac{d}{dx}[(\frac{1}{n}-1)P_{n-2}(x) + x(2-\frac{1}{n})P_{n-1}(x)]$$

Differetiating this function once yields once for n=1, 2, 3, etc. yields-

$$\frac{dF(1,x)}{dx} = \frac{d}{dx}(1-x^2)\frac{dP_n(x)}{dx} = -2P_1(x)$$

$$\frac{dF(2,x)}{dx} = \frac{d}{dx}(1-x^2)\frac{dP_2(x)}{dx} = -6P_2(x)$$

and-

$$\frac{dF(3,x)}{dx} = \frac{d}{dx}(1-x^2)\frac{dP_3(x)}{dx} = -12P_3(x)$$

From these results one can conclude that the differential equation governing the Legendre polynomials is-

$$\frac{d}{dx}(1-x^2)\frac{dP_n(x)}{dx} + n(n+1)P_n(x) = 0$$

If we now multiply this equation by $P_m(x)$ and integrate over the range -1<x<+1, one finds-

$$\int_{x=-1}^{+1}(1-x^2)\frac{dP_n(x)}{dx}\frac{dP_m(x)}{dx}dx = n(n+1)\int_{x=-1}^{1}P_n(x)P_m(x)dx$$

One notes at once that if integer n differs from integer m, that the left side of this equality vanishes. Thus for [n,m]=[1,3] and [2,4] we have-

$$\int_{x=-1}^{+1}(1-x^2)(3x) = 0 \quad and \quad \frac{3}{2}\int_{x=-1}^{+1}[(1-x^2)(35x^4-15x^2)]dx = 0$$

When n=m, however, the left side does not vanish. For n=m=1, 2, 3, etc we find

$$\frac{1}{n(n+1)}\int_{x=-1}^{+1}(1-x^2)\frac{dP_n(x)}{dx}\frac{dP_m(x)}{dx}dx = \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, etc$$

That is, the Legendre polynomials are orthogonal in the range -1<x<+1, with-

$$\int_{x=-1}^{+1}P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{nm}$$

Here δ_{nm} is the Kronecker delta with value +1 when n=m and 0 when n≠m.

The fact that the $P_n(x)$s are orthogonal allows us to expand any function f(x) in terms of them. One has-

$$f(x)=\sum_{n=0}^{\infty}C_nP_n(x) \quad with \quad C_n=\frac{2n+1}{2}\int_{x=-1}^{+1}f(x)P_n(x)dx$$

For f(x)=x^4, this expansion yields the three term series-

$$x^4=\frac{1}{35}[7P_0(x)+20P_2(x)+8P_4(x)]$$

Note by setting 2C$_n$/(2n+1)=F[n] we obtain the Legendre transform pair-

$$F(n)=\int_{x=-1}^{+1}f(x)P_n(x)dx \quad with \quad its \quad inverse \quad f(x)=\sum_{n=0}^{\infty}[\frac{2n+1}{2}]F(n)P_n(x)$$

Unlike Laplace transforms, Legendre transforms are generally much more complicated. One of the simpler ones occurs for the Dirac delta function. There

$$F(n)=\begin{cases} 0\ ,\ n\ odd \\ \dfrac{n!(-1)^{n/2}}{2^n[\Gamma(0.5n+1)]^2}\ ,\ n\ even \end{cases}$$

On inverting, it produces the identity-

$$\delta(x)=\sum_{n=0}^{\infty}[\frac{(4n+1)}{2}][\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}]P_{2n}(x)$$
$$=\frac{1}{2}P_0(x)-\frac{5}{4}P_2(x)+\frac{27}{16}P_4(x)-\frac{65}{32}P_6(x)+\frac{595}{256}P_8(x)-...$$

It takes about the first eight terms in this series to start seeing a function approaching infinity at x=0 and vanishing for all other x in -1<x<+1.

There are numerous integrals involving Pn(x). One of the simplest is-

$$\int_{x=-1}^{+1}x^nP_n(x)dx=\frac{2(n!)}{1(3)(5)(..)(2n+1)}=\frac{(n!)^2\,2^{n+1}}{(2n+1)!}$$

as can be established by looking at the values for n=0,1,2,3,4. Another integral is-

$$\int_{x=-1}^{+1} \frac{P_n(x)}{\sqrt{1-x}} dx = \frac{2\sqrt{2}}{(2n+1)}$$

which can be established by noting the integral has values $2^{1.5}/3$, $2^{1.5}/5$, $2^{1.5}/7$ for n=1,2,3, respectively. Another interesting integral is-

$$\int_{x=-1}^{+1} \frac{P_{2n}(x)}{(1+x^2)} dx = A_n + B_n \pi$$

where the values of A_n and B_n increase in magnitude with increasing n and have opposite signs. We find [An, Bn]=[3, -1] for n=1 and for n=10 take on the values-

A_{10} = -7968815657984/969969 and B_{10}=+1338924271/512

We thus find the interesting equality-

$$\pi = \frac{4080033616887808}{1298715036217599} + \frac{1024}{1338924271} \int_{x=0}^{+1} \frac{P_{20}(x)}{(1+x^2)} dx$$

where the first term already approximates π to some 14 places and the second term adds a correction of order 10^{-14}. Since the second term in this expression for π becomes progressively smaller with increasing n, it is clear that-

$$\pi = \lim_{n \to \infty} \left[-\frac{A_n}{B_n} \right]$$

A 38 place accurate approximation for π is found at n=25 and is given by the quotient-

$$\left[\frac{-A_{25}}{B_{25}} \right] = \frac{64534406708870849908412875242673384126519705 6}{20541939654439137369384796202222344020235192 5}$$

We can also generalize the above integral to find-

$$\int_{x=-1}^{+1} \frac{P_{2n}(x)}{k^2 + x^2} dx = N_n + M_n \arctan(\frac{1}{k})$$

Where N_n and M_n are numbers depending on n. Again one finds that an approximation for arctan(1/k) will be given by-

$$\arctan\left[\frac{1}{k}\right] \approx \left[\frac{-N_n}{M_n}\right]$$

The convergence rate of this quotient with increasing n is found to be quite rapid. For n=10 one already finds arctan(1/5) given accurately to 40 places.

A good approximations for ln(2) can be gotten from the evaluation -

$$\int_{x=-1}^{+1} \frac{P_n(x)}{(3+x)} dx = S_n + T_n \ln(2)$$

For large n the term on the left divided by T_n becomes small compared to the quotient S_n/T_n. Thus at n=20 we find the 30 digit accurate result-

$$\ln(2) \approx \frac{46712468052304621 7643}{(2586584)(260543813797441)}$$ **=0.693147180559945309417232121458..**

Finally let us look at an integral representation for the Legendre polynomials. Using the second order differential equation for $P_n(x)$ and an Euler Kernel $(x-t)^{(n+1)}$as the starting point, one finds(see our class notes in the above ODE course) that-

$$P_n(x) = \frac{1}{2\pi i} \oint \frac{(t^2-1)^n}{2^n (t-x)^{n+1}} dt$$

This result is known as the Schlaefli Integral. By contour integration around the (n+1) order pole at x, this integral can be evaluated by the Cauchy theorem to yield-

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

This represents the famous Rodriques derivative formula for generating Legendre polynomials. It is not quite as convenient for finding values of $P_n(x)$ at large n via computer as is the generating formula given earlier.

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