MORE ON PRIME NUMBERS

It is known that all prime numbers, with the exception of 2, must take the form of an odd number N=2n+1, n=0,1,2,3,…However it is only for special values of n that N is a prime. A convenient way to present these primes is by the following four column table-

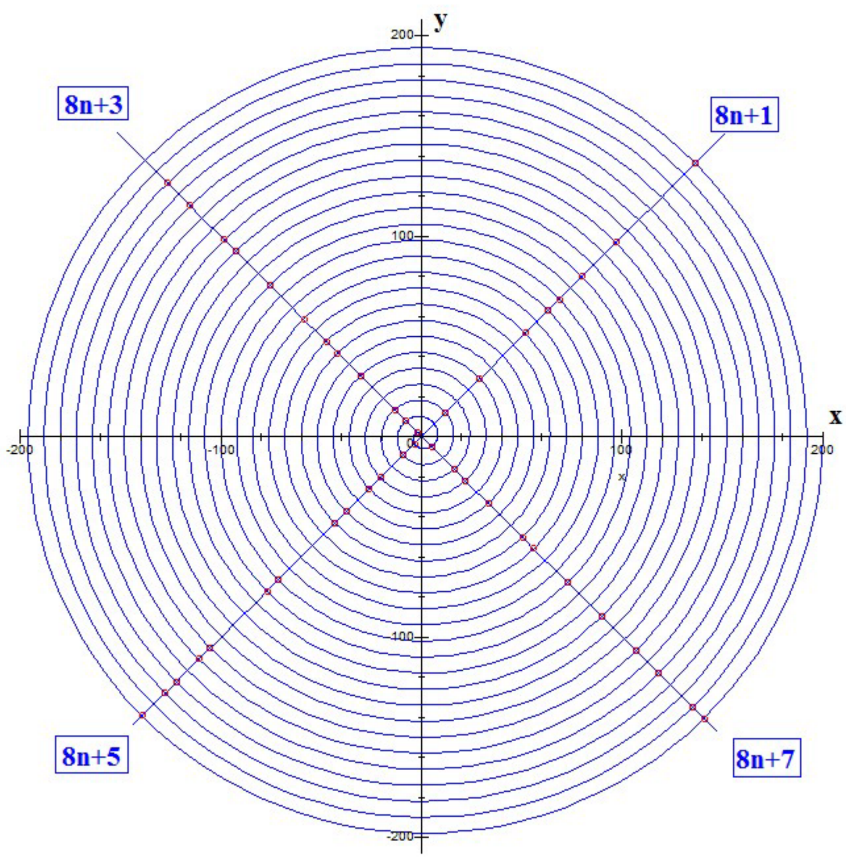
n	8n+1	8n+3	8n+5	8n+7
0	-	3	5	7
1	-	11	13	-
2	17	19	-	23
3	-	-	29	31
4	-	-	37	-
5	41	43	-	47
6	-	-	53	-
7	-	59	61	-
8	-	67	-	71
9	73	-	-	79
10	-	83	-	-
11	89	-	-	-
12	97	-	101	103
13	-	107	109	-
14	113	-	-	-
15	-	-	-	127
16	-	131	-	-
17	137	139	-	-
18	-	-	149	151
19	-	-	157	-
20	-	163	-	167
21	-	-	173	-
22	-	179	181	-
23	-	-	-	191
24	193	-	197	199

You will note that the primes in a given column differ from each other by multiples of eight and that neighboring primes at n differ from those at n+1 by six. Thus, for example, at n=12 the term 8n+5=101 differs from the n=13 term of 8n+3=107 by six as does the n=14 term of 113 differ by six from 107. The term 8(13)+5=109 differs from 8(18)+5=149 by 5(8)=40. The dashes represent odd numbers which are not prime and hence composite. Our reason for representing all primes by the four columns above is that several years ago we found a convenient a way to plot all integers in such a manner that all odd numbers lie along the diagonals y=x and y=-x and all even numbers along the x or y axis in the x-y plane. The odd numbers , and hence all prime numbers(except N=2), are found at the intersections of the Archimedes Spiral –

$$\sqrt{x^2+y^2} = \frac{4}{\pi}\arctan(\frac{y}{x}) \text{ and the diagonals } y = \pm x$$

We show you the pattern in the following graph for the 45 prime numbers given in the above table-

LOCATION OF THE FIRST 45 PRIME NUMBERS IN THE X-Y PLANE



Note that this method of ordering the primes is far superior to that given by the well known Ulam Spiral. Many computer programs have been written for the location of primes along the Ulam Spiral and some investigators even believe that the resultant patterns give insight into the properties of primes. However, as we showed earlier (see our article on the Morphing of Ulam) the Ulam Spiral pattern can be reduced to the much simpler pattern for primes given above by a simple transformation. It means, of course, that the pattern found for Ulam's Spiral is really no more than a statement in disguised form that all odd numbers (be they prime or composite) will lie along the diagonals shown above.

To determine whether or not an intersection point between the Archimedes Spiral and the 45 degree diagonal lines yields a prime number will require applying at least one among many possible prime tests . The names of Erastothenes, Fermat, Lucas and Lehmer are attached to these different tests. Consider the number N=81=8(10)+1 which lies along the diagonal in the first quadrant of the x-y plane. To determine if it prime or not one can apply one of the simplest prime tests, namely that of the Sieve of Erastothenes. Here one takes all rime numbers from 3 through sqrt(N) and divides it into N. If any of the quotients turn out to be integer, then N is composite. For N=81 one finds 81/3=27 and hence 81 is composite. Consider next the larger number N=199=8(24)+7 lying along the diagonal in the fourth quadrant. Here one needs to divide 199 in turn by 3, 5, 7, 11, and 13 . This yields 199/3=66.333, 199/5=39.800, 199/7=28.428, 199/11=18.090, and 1999/13=6.993. Thus N=199 must be prime since none of the prime divisors yield an integer value.

When the number N gets large this process becomes very time consuming and is the reason codes in cryptography using public keys are relatively secure. From the Prime Number Theorem one knows that the number of primes less than N equals approximately N/ln(N) provided that N is large. Thus the approximate number of primes n_p lying between N_2 and N_1 equals-

$$n_p = \frac{N_2}{\ln(N_2)} - \frac{N_1}{\ln(N_1)} \quad , \quad N_1 < N$$

Take the number of primes lying between N_1=10,000 and N_2=10,100 . The predicted number by the above formula is n_p=9.7. Using the one line MAPLE computer program-

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for n from 0 to 100 do {n, isprime(10,000+n)}od;
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we find the following ten primes-

10,007 10,009 10,037 10,039 10,061 10,067 10,069 10,079 10,091 10,099

in the same range. The approximation is thus seen to be very good.

Another test for primes is the use of Fermat's Little Theorem which states that a number N is **likely** to be prime if-

$$\frac{[a^{N-1} - 1]}{N} = INTEGER\ where\ base\ 'a'\ equals\ 2,3,4,....$$

We can get rid of the -1 in the expression by rewriting things as-

$$\frac{[3\cdot 4^{N}+6\cdot 2^{N}-8\cdot 3^{N}]}{12N}=Integer$$

or-

$$\frac{[2^{N}(2^{N}-2)]}{4N}=Integer$$

Testing the prime N=7 yields the integer values 386 and 576, respectively , in the last two expressions. Thus N=7 is prime. That N=9 is composite follows by noting the above integer quotients yield the non-integer results 5852.22222.. and 725.33333…. The difficulty with the Fermat Prime Test and its above combinations is that the numbers become unwieldy as N gets large. For example, to show that N=199 is prime requires an evaluation of-

$$2^{197}(2^{199}-2)/(199)$$

This indeed turn out to be an integer and hence N=199 is prime. However, the integer one is dealing with is the 117 digit monster-

81100812753985822539444697613159920676184227161721843995937591984\5\
625184062732106401154519041999919165806545477828608

What is quite clear from these results is that for larger primes the Fermat test and its variations (and also the Sieve of Eratothenes) become impractical as N gets large.

Be that as it may, we can determine, at least in principle, which odd number N along a given diagonal is prime by using the Fermat Test in the form-

$$\frac{[2^{k}\cdot 256^{n}-1]}{[N]}=Integer\ with\ k=0,1,2,3\ and\ N=8n+k+1$$

The value of k used depends on which diagonal the number N lies. One has k=0 for quadrant one numbers, k=1 for quadrant two numbers, k=2 for quadrant three numbers and k=3 for quadrant four numbers. To test things out, look at N=17 which lies along the diagonal in the first quadrant so that k=0 and n=2. We get-

$$\frac{[256^{2}-1]}{17}=\frac{65535}{17}=3855\quad so\ N=17\ is\ prime$$

As a second example take the number N=139 from the 8n+3 column at n=17. The Fermat test gives-

$$\frac{[2\cdot 256^{17}-1]}{139}=250682837213698551501075969791460898 2237$$

Thus N=139 is also a prime number.

We point out that the well known Mersenne Primes of the form M(m)=2^m-1 for m=2,3,5,7,13,17,19,31,61,89, etc all lie along the diagonal in the fourth quadrant so that M(n) has to be of the form 8n+7. Thus we get the identity-

$$2^{m-3}=n+1$$

Consider the Mersenne prime M(7)=127. Here the identity yields n=15 meaning the number lies in the fourth quadrant at the 15th turn of the Archimeds spiral.

Note that so far only 47 Mersenne Primes have been found although it is believed there are an infinite number. They form a small sub-set of the remaining infinite number of 8n+7 primes. Look at column four of the above table to see the first twelve of these. In binary form all primes and composite numbers lying along the diagonal in the fourth quadrant end in …111 . Thus –

167= 10100111 , 191=10111111,

and the Mersenne Prime –

2^{19}-1= 524287 becomes 1111111111111111111 in binary

and-

2^{61}-1= 2305843009213693951 becomes

1111111111111111111111111111111111111111111111111111111111111

We point out that the Mersenne Numbers 2^n-1 which are not prime still have the same 111111111… binary notation as this is a consequence of their definition equation and says nothing about the number being prime or not. As a final observation we note that all known Mersenne Primes end in 1 or 7 when expressed in base10 notation. Thus those primes of the form 8n+7 ending in 3 or 9 may be discarded as possible Mersenne Primes.

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