EXAMINATION OF THE INTEGRAL $F(N)=\int_{x=1}^{\infty}\prod_{n=1}^{N}\frac{1}{(n^2+x^2)}dx$

In some recent discussions on the AGM method we looked at integrals of the type-

$$I(a,b)=\int_{x=0}^{\infty}\frac{dx}{\sqrt{(a^2+x^2)(b^2+x^2)}}=\int_{t=0}^{\infty}\frac{dt}{M^2+t^2}=\frac{\pi}{2M}$$

where M was the arithmetic-geometric mean of a and b. On making some naïve attempts to extend this approach to other powers of the term in the denominator it became clear that the AGM fails to work (with a few exceptions such as for the complete elliptic integral of the second kind) for general integrals of the type-

$$J(a,b,p,q)=\int_{t=0}^{\infty}\frac{dx}{(a^2+x^2)^p(b^2+x^2)^q}$$

This does not mean that this last integral does not have closed form solutions such as -

$$\int_{t=0}^{\infty}\frac{dx}{\left[\sqrt{1+x^2}\right]\left(\frac{1}{2}+x^2\right)}dx=\frac{\pi}{2}$$

and-

$$J(a,b,1,1)=\int_{x=0}^{\infty}\frac{dx}{(a^2+x^2)(b^2+x^2)}=\frac{\pi}{2ab(a+b)}$$

You will notice in this last integral that $ab=G^2$ and $(a+b)/2=\bar{x}$, where $G=\text{sqrt}(ab)$ is the geometric mean and $\bar{x}=(a+b)/2$ the arithmetic mean. In trying to generalize this last integral we came up with the convergent definite integral-

$$K(N)=\int_{x=0}^{\infty}\frac{dx}{(N^2+x^2)(\ldots)(1^1+x^2)}=\int_{x=0}^{\infty}\frac{dx}{\prod_{n=1}^{\infty}(n^2+x^2)}$$

Evaluating this integral for N=1, 2, 3,… , we find-

K(1)=π/2, K(2)=π/12 , K(3)=π/120, K(4)=π/2016, K(5)=π/51840, and K(6)=π/1900800

These results can be rewritten, using the nth power of the geometric mean $G[n]^n=1\cdot2\cdot3\cdot\cdot n=n!$, as-

$$K(2) = \frac{\frac{\pi}{2}}{2!\,(3)} \;,$$

$$K(3) = \frac{\frac{\pi}{2}}{3!\,(10)} \;, \qquad K(4) = \frac{\frac{\pi}{2}}{4!\,(42)} \;, \qquad K(5) = \frac{\frac{\pi}{2}}{5!\,(216)} \;, \qquad K(6) = \frac{\frac{\pi}{2}}{6!\,(1320)}$$

If one now examines the sequence [3, 10, 42, 216, 1320] one sees that 3x3+1!=10, 4x10+2!=42,

5x42+3!=216 and 6x216+4!=1320. We can thus say that the nth element p(n) of this sequence equals –

$$p(n) = n\,p(n-1) + (n-2)!$$

Thus p(7)=7(1320)+5!=9360 and p(8)=8(9360)+6!=75600. Thus we can state that-

$$\int_{x=0}^{\infty} \frac{dx}{\prod_{n=1}^{8}(n^2 + x^2)} = \frac{\pi}{6096384000}$$

You can actually carry out the integration required by this last lengthy integral to find perfect agreement.

Another variation on the integral K(N) is-

$$L(N) = \int_{x=0}^{\infty} \frac{dx}{(a^2 + x^2)^N} = \frac{1}{a^{N-1}} \int_{z=0}^{\infty} \frac{dz}{(1 + z^2)^N} \quad where \;\; z = \frac{x}{a}$$

With the additional transformation z=sqrt(U)/sqrt(1-U) and using the identities Γ(n+1)=nΓ(n) and Γ(1/2)=sqrt(π), we find-

$$L(1) = \frac{\pi}{2} \;, \qquad L(2) = \frac{\pi}{4a}, \qquad L(3) = \frac{3\pi}{16a^2} \;, \qquad L(4) = \frac{5\pi}{32a^3} \;, \qquad L(5) = \frac{35\pi}{256a^5}$$

Notice that the values of the above integrals K(N) and L(N) are proportional to the first power of π. So if we pick some other definite integral where this proportionality does not go as the first power, we have a way to express π as the quotient of two integrals. Thus , for example, one has-

$$\sqrt{\pi} = \frac{\int_0^{\infty} \frac{dx}{\left(1 + x^2\right)^2}}{\int_0^{\infty} x^2 e^{-x^2} dx} = 1.77245\ldots$$

What is clear from all these examples is that AGM method does not generally apply to integrals other than the integral I(a,b) given at the beginning of the discussion. In particular we found that-

$$\int_{x=0}^{\infty} \frac{dx}{\left(x^2+1\right)\left(x^2+4\right)} \quad \text{is not equal to} \quad \int_{t=0}^{\infty} \frac{dt}{\left(M^2+t^2\right)^2}$$

where the AGM of 1 and 2 is M=1.4567910… . Rather the value of M should be $3^{1/3}$=1.4422495… for equality between these two integrals.

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