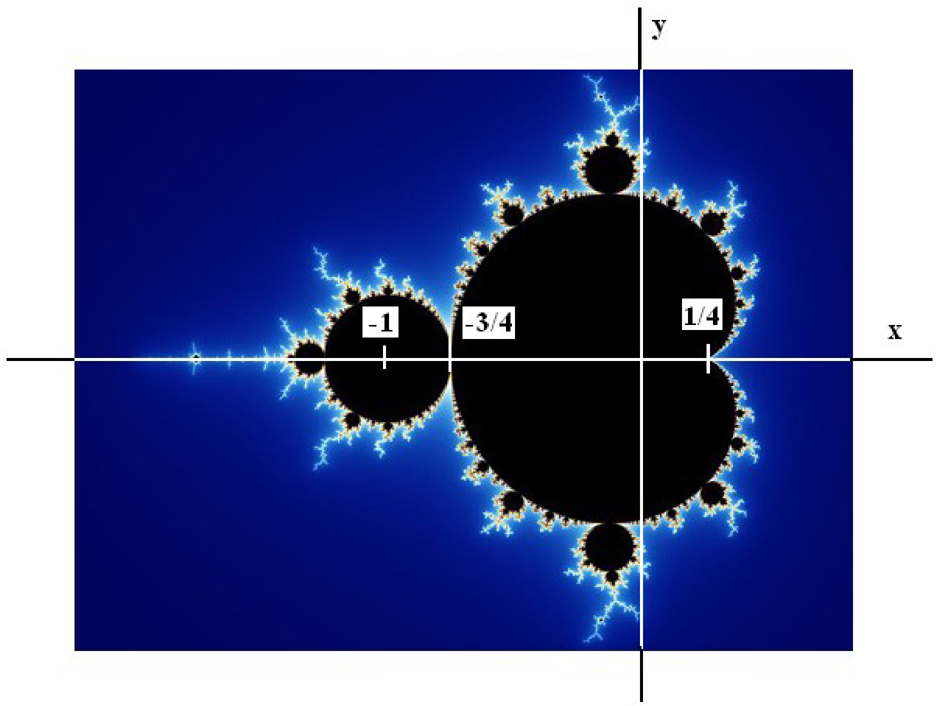
PATHS TO CONVERGENCE FOR THE MANDELBROT SET

Several years ago while teaching an undergraduate class on complex variables(see http://www.mae.ufl.edu/~uhk/ANALYSIS.htm) I noticed that the iteration A[n+1]=I^A[n] subject to A[0]=0 showed an interesting three spiral path to a convergence point A[inf]=0.43828..+I 0.36059… in the complex plane. Different, but still interesting convergence patterns also exist when the initial starting point A[0] is changed and a complex constant c=x=iy is added to the iteration formula. In thinking more about this behavior , I recalled the intricate figures which are produced with the Mandelbrot Set A[n+1]=(A[n])^2+c subject to A[0]=0. In the Mandelbrot case researchers have concentrated essentially on the rate at which the iteration blows up to produce interesting and quite intricate colored graphs such as that shown here-

MANDELBROT FRACTAL A[n+1]=(A[n])^2+x+iy WITH A[0]=0
(black indicates convergence)



(adaptation of figure by Wolfgang Beyer in Wikipedia)

They have not concentrated on the path to convergence of A[n] in the complex plane in the black regions where the iteration is bounded. It is in these regions, confined to a good approximation within a cardioid and circle, that paths to convergence will be found , just as we found them for A[n+1]=I^A[n]. Our purpose here is to determine some of these convergence paths for the Mandelbrot Set.

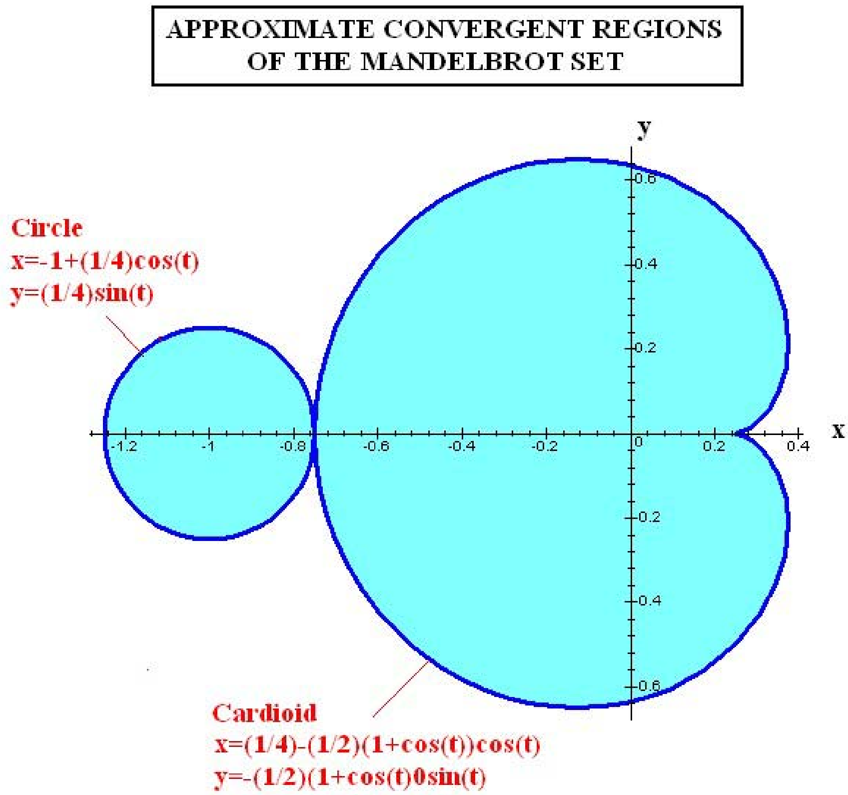
We begin this study by confining ourselves to the two regions given implicitly by-

$$x=\frac{1}{4}-\frac{1}{2}\{1+\cos(t)\}\cos(t),\ \ y=-\frac{1}{2}\{1+\cos(t)\}\sin(t)$$

and

$$x=-1+\frac{1}{4}\cos(t),\ \ y=\frac{1}{4}\sin(t)$$

A graph of these regions , showing the values of c=x+iy for which things are guaranteed to converge to a unique value (or are at least bounded), follows-



A few points beyond this blue shaded region also will converge, however, most correspond to points inside. If we now use the two line MAPLE program-

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A[0]:=0: for n from 0 to 50 do A[n+1]:=evalf(A[n]^2+x+y*I, 8)od:

with(plots): pointplot({seq([Re(A[n]),Im(A[n])],n=0..50)},symbol=circle,
color=blue, scaling=constrained);
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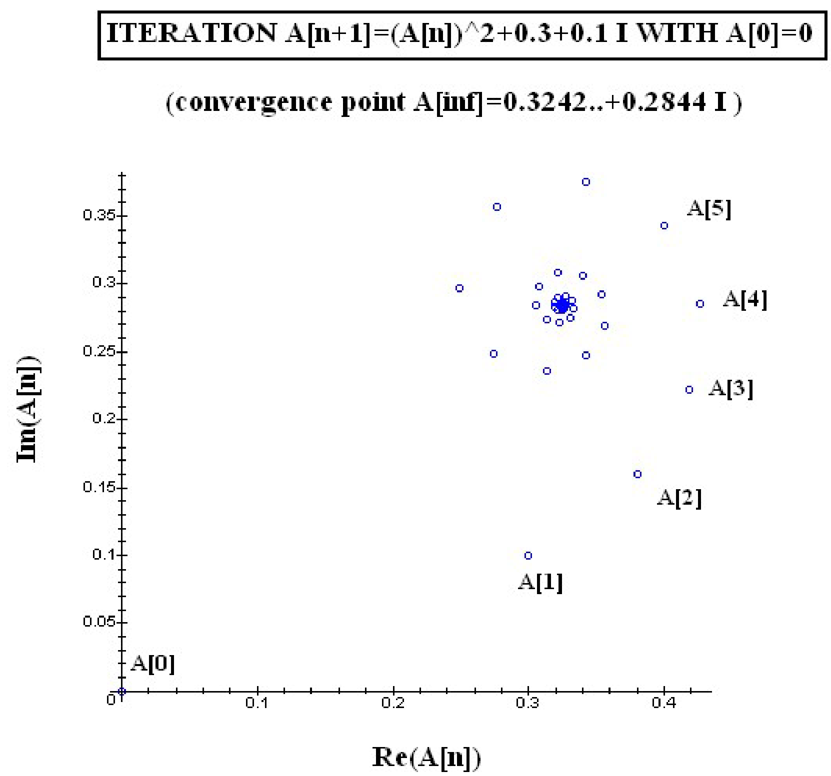
chosing a point c=x+iy within the light blue region or just slightly beyond it, the result of the first line will indicate convergence. Next, the second line will graph the A[n] paths to a convergence point A[∞] . Let us demonstrate for c=0.3+0.1 I with

A[0]=0. This point lies within the cardioid not far from its cusp. The path to convergence looks like this-



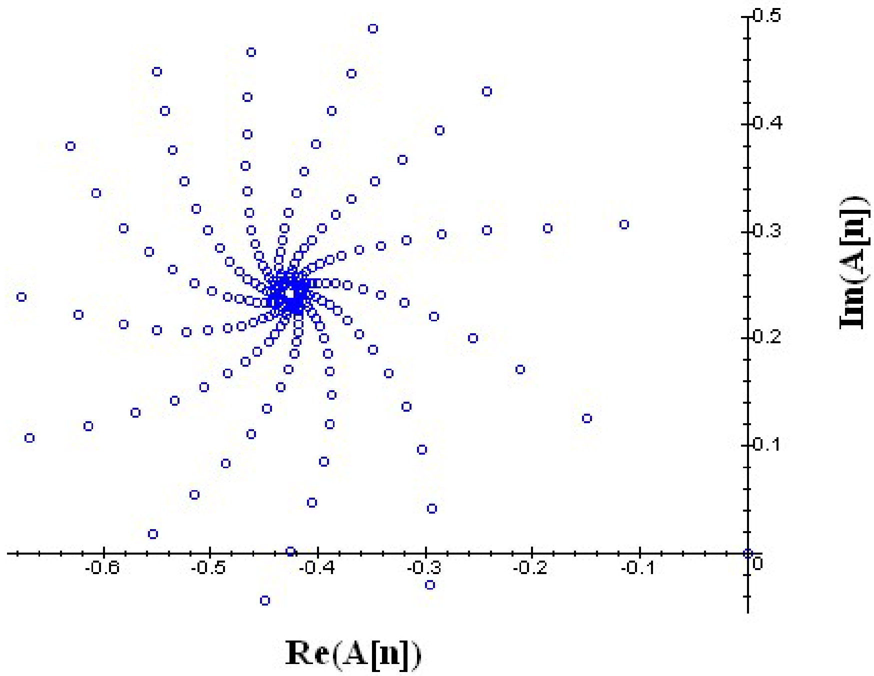
The path is seen to be a point spiral and its convergence point A[∞] can be determined by the quadratic equation-

$$A[\infty]=A[\infty]^2+c \text{ and its solution } A[\infty]=\frac{1}{2}-\sqrt{\frac{1}{4}-c}=0.3242..+0.2844I$$

This final convergence point does not depend on the initial condition A[0] although the spiral path to A[∞] will differ. The most interesting convergence paths are found for values of c=x+iy near the convergence border.

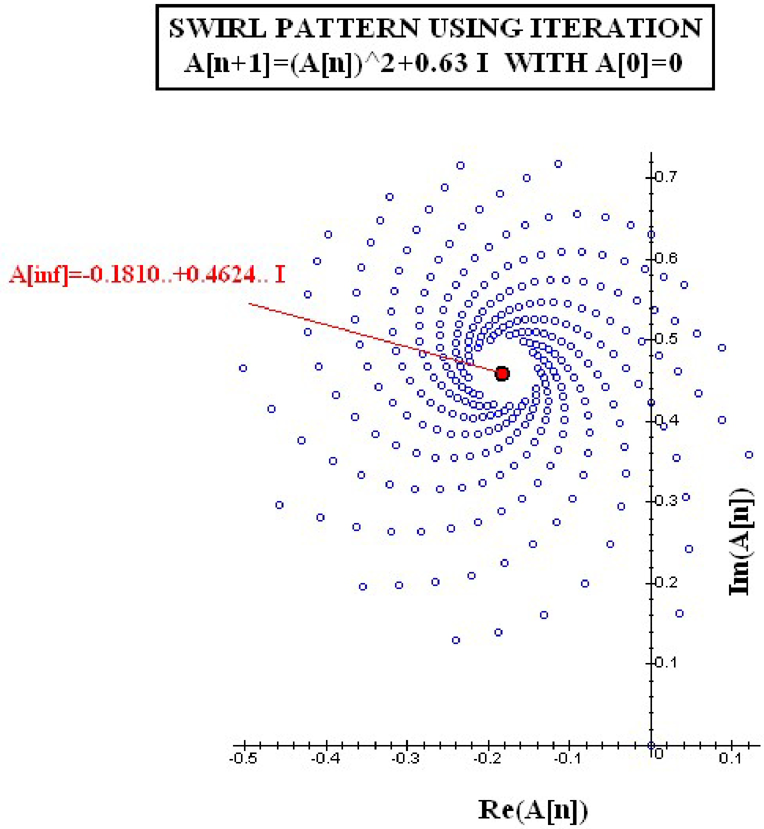
Another pattern is found for c=-0.55+0.45 I. Here one has the twelve armed spiral (reminiscent of a lawn sprinkler) shown-

PATH TO CONVERGENCE FOR c=-0.55+0.45 I WITH A[0]=0



with A[∞]=0.5-sqrt(0.8-0.45 I)=-0.42678..+0.24277..I. Note that the progression A[1], A[2], A[3], etc does not here follow along a single spiral arm but rather jumps from one arm to the next.

For a third A[n] pattern we choose the case for c=0.63 and A[0]=0. This produces the slowly converging 300 point swirl figure-



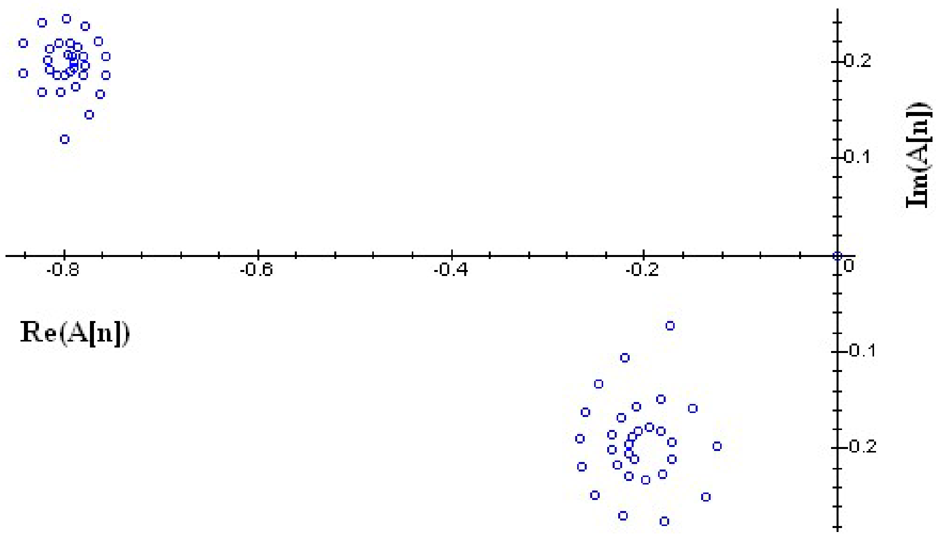
Eventually it will converge to the indicated red point-

$$A[\infty] = \frac{1}{2} - \sqrt{\frac{1}{4} - 0.63I} = -0.18109.. + 0.462488..I$$ -

The reason for the slow convergence in this case is that we are nearly on the cardioid boundary.

Finally let us look at the convergence path for the case c=-0.8+0.12 I . Here c lies within the small circle within the complex c plane. The iterated path consists of two spirals as is shown here-

ITERATION A[n+1]=(A[n])^2-0.8+0.12 I WITH A[0]=0
LEADING TO TWO SPIRALS



This time our A[inf] criterion fails since one does not reach a unique limit and rather A[n] and A[n+1] lie on different spirals. That is, one does not converge to a unique number, but rather one is dealing with a case where the iteration continues to jump between two bounded values of approximately A[inf]=-0.8+0.2I and A[inf]=-0.2-0.2I. The average of these numbers seems to match the quadratic formula prediction of -0.52636..+0.058458..I .

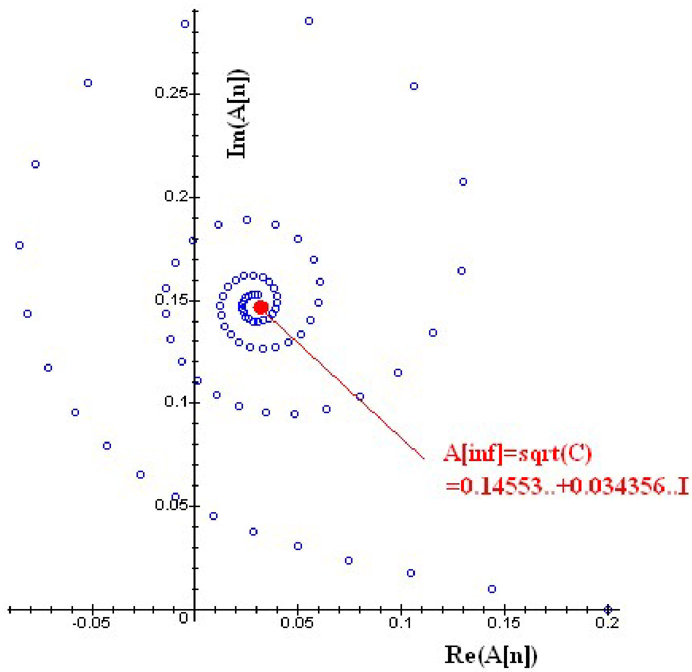
We should point out that the spiral curves found in the above iteration examples involving the Mandelbrot Set are expected to continue to be found for an infinite number of other iterations of the form-

$$A[n+1] = F(A[n]) + x + Iy$$

Indeed , as already mentioned earlier, we first ran into such spiral curves when carrying out the iteration A[n+1]=exp{(Iπ/2)A[n]}+0 with A[0]=0. An example of

such an additional iteration is found for F=A[n](1-A[n]) subject to A[0]=0.2 and C=x+Iy=0.02+I 0.1. Here is the spiral path toward the convergence point obtained–

SPIRAL GENERATED BY THE ITERATION A[n+1]=A[n](1-A[n])+C
SUBJECT TO A[0]=0.2 AND C=0.02+0.01 I



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