PRIME NUMBERS GENERATED BY $[2^{2n+1}+1]/3$, $[2^{2n+1}-3]/5$, and $[2^{2n+1}+3]/5$

In an earlier note we discussed the occurrence of prime numbers along four diagonals in a plane were all positive integers occur at the intersections of an Archimedes spiral and the lines x=0, y=0, x=y and y=-x. There appear to be four basic classes of odd numbers and hence sub-classes of primes defined as 8n+1, 8n+3, 8n-1, and 8n-3 for integer n. A special subset of 8n-1 are the well known Mersenne numbers $M(n)=2^{2n+1}-1$ which all lie along the diagonal in the fourth quadrant and are prime for n=1,2,3,6,8 . In examining the subset S1(n)=2^(2n+1)+1 of the odd integers 8n+1 lying along the diagonal in the first quadrant we noted that when these are divided by 3 one often finds prime numbers. These numbers are known in the literature as Wagstaff primes(see http://en.wikipedia.org/wiki/Wagstaff_prime)and can be generated by the simple one line program -

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for n from 1 to 21 do {n,(2^(2*n+1)+1)/3, ifactor(2^(2*n+1)+1)/3}od;
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which yields the 21 results-

n	W(n)=(2^(2n+1)+1)/3
1	3= [8(0)+3]
2	11= [8(1)+3]
3	43= [8(5)+3]
4	171= [8(0)+3] · [8(0)+3] · [8(2)+3]
5	683= [8(85)+3]
6	2731= [8(341)+3]
7	10923= [8(0)+3] · [[8(1)+3] · [8(41)+3]
8	43691= [8(5461)+3]
9	174763= [8(21845)+3]
10	699051= [8(0)+3] · [8(5)+3] · [8(677)+3]
11	2796203= [8(349525)+3]
12	11184811= [8(1)+3] · [8(31)+3] · [8(506)+3]
13	44739243= [8(2)+3] · [8(3)+3] · [8(10901)+3]
14	178956971=[8(7)+3] · [3033169]
15	715827883=[8(89478485)+3]
16	2863311531=[8(0)+3] · [8(8)+3] · [8(85)+3] · [20857]
17	11453246123=[8(1)+3] · [8(5)+3] · [281] · [86171]
18	45812984491=[8(3222635)+3] · [1777]
19	183251937963=[8(0)+3] · [8(341)+3] · [2236689]
20	733007751851=[8(10)+3] · [8 (11039277337)+3]
21	2932031007403=[8(366503875925)+3]

We note that these numbers are rich in primes(indicated in red) and that they all lie along the diagonal in the second quadrant characterized by the factor [8(m)+3] . A generating formula is –

$$W(n+1) = 4W(n) - 1 \;\; starting \; with \;\; W(1) = 3$$

from which follows -

$$W(n+m) = 4^m W(n) - \sum_{k=0}^{m-1} 4^k = \left[(2^{2(n+m)+1} + 1)/3\right]$$

In binary form the numbers W(n) reading from left to right start with (n-1) pairs of 10 and end in 11. Thus W(30)= (2^61+1)/3 =768614336404564651 reads-

101010101010101010101010101010101010101010101010101010101011

The fact that the binary form of all of the W(n)s likewise end in 001 means(as already noted above) that they all lie along the diagonal in the second quadrant. The values of n for which W(n) is found to be a prime number are –

n=1,2,3,5,6,8,9,11,15,21,30,39,50,63,83,95,99,156,173,350,854,…

and so(as with Mersenne primes) the density of primes becomes sparser as n increases. This is consistent with the Prime Number Theorem. The prime corresponding to n=854 equals the large 514 digit number-

W(854)=9619252724870106900591857521914969755994617926162485749828680263
442652106934332647419115450350548509594113720023679578428867652599441500
099408187466188831829627322561298620142064115434370067980896141705699219
480691615366345077753886654052361984948213632109502554264973588138936426
510951217614006613784979416372970841528398839021290713797778493018446941
241754769857894878843499740605322478881600054025752566680081461168156277
505142145985544055010083285836939763680507287209419644373726922686975212
036857299070528171

Note that all W(n)s end in either 1 or 3 so that W(n)-1 will be an even number ending in zero or two. Compare this result to the Mersenne numbers M(n)=2^(2n+1)-1 which all end in either 1 or 7. One finds that-

$$W(n) = \left[\frac{M(n)+2}{3}\right]$$

This equality does not, however imply that a prime W(n) corresponds to a prime M(m) . For example W(5)and W(21) are prime while M(5)and M(21)are not. Special cases also exist where both W(n) and M(n) are prime. Examples are found for n=1, 2, 3, 6, 8, 9, 15 and 30.

The simplest way to determine the primality of W(n) is to use the MAPLE command- isprime(W(n)) or ifactor(W(n)) which are based a Lucas and other primality tests. It is also possible to use a modified sieve of Eratosthenes approach which here works by dividing W(n) by 8m+3 for m up to values yielding a number equal to the square root of W(n). If any integer values of this quotient are found then the number is composite. In case no integer quotients are discovered by dividing by 8m+3, one can conclude that the number is prime without the need to divide by the other odd numbers 8m-1, 8m+1 and 8m-3 (at least that is what the results of the above table indicate). Consider the case W(10)= 699051 where sqrt(W(10))=836.09.. and the maximum n needed for the test will be n=104, however, as the table shows, one usually has to consider only the lower values of m. Carrying out the one line program-

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for n from 1 to 10 do {n,evalf(699051/((8*n+3)))}od;
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we find the integer value 16257 for the quotient at m=5. Hence one can write W(10)=699051=43(16257)=[8(5)+3]*16257 which means W(10) is composite and not prime. This approach, unfortunately becomes somewhat unwieldy when dealing with very large W(n). In testing the twelve digit number W(20)=733007751851 one would generally need to use integer values up to m=10,702. Fortunately, this number is shown to be composite using the relatively small number m=10. We find that W(20)=[8(10)+3]*[8831418697] and thus it takes only some ten divisions starting with m=0 to establish this fact. To establish the primality of the next number W(21), however, would take several thousand divisions when using the sieve of Eratosthenes approach.

Some other subsets for primes which we have recently found and are investigating in more detail are K(n)=[2^(2n+1)-3]/5 and L(n)=[2^(2n+1)+3]/5. We find K(n) is prime for n=1,5,7,11,17,35,47,… and L(n) prime for n=2,4,12,22,30,.. . In binary the K(n)s end in 001 and hence lie along the diagonal in the first quadrant while the L(n)s end in 111 and thus lie along the diagonal in the fourth quadrant. Two examples of these primes are-

L(30)=(2^30+3)/5=461168601842738791

and

K(47)=(2^95-3)/5=7922816251426433759354395033

Both K(n) and L(n) are related to the Mersenne Numbers through the equalities-

$$K(n)=\frac{1}{5}\left[M(n)-2\right] \quad and \quad L(n)=\frac{1}{5}\left[M(n)+4\right]$$

Again prime M(n) does not necessarily mean either K(n) or L(n) are prime or vice-versa. We also have the recurrence relations-

$$K(n+1)=[4K(n)-(9/5)] \quad and \quad L(n+1)=[4L(n)-(9/5)\}$$
$$with \quad K(1)=1 \quad and \quad L(2)=7$$

which rapidly yield prime numbers such as-

K(47) =[2^{95}-3]/5= 7922816251426433759354395033

reading-

**1100110011001100110011001100110011001100110011001100110011001100110011001100
1100110011001100110011001**
in binary and-

L(30)=[2^{61} +3]/5= 461168601842738791

**which reads- 11001100110011001100110011001100110011001100110011001100111
in binary.**

We can generalize the above results by noting that a large number of primes can be produced by the number-

$$N^{\pm}(n,m)=\frac{[2^{2n+1}\pm(2m-1)]}{2m+1}$$

The first few of these for N^{+} when n=m are-

n	N^{+}(n,n)
1	3
2	7
3	19
6	631
26	169947155749831
33	2202596307308603179
41	116522970565265462622283

Also it is noted that one need not stay with functions involving $2^{(2n+1)}$ to generate primes. As already known to Euler, certain polynomials can be used to generate primes. For example, Euler's own polynomial n^2-n+41 will generate 40 correct primes 41, 43, 47,… through 1601 . After that it fails every once in a while such as for values generated with n=41, 42, 45, 50,57, 66,77, 82, 83, 85, 88, 90, 97,…Other polynomial forms have been discovered of which the best is a quintic in n which yields 57 consecutive primes. It appears that there will never be a single polynomial formula capable of generating all primes. This does not mean, however, that existing formulas are useless, since they can still be used to generate large primes. For example using the Euler formula for n=1234567890+1 yields the prime 1524157876253620031.

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