

SUMMATION OF THE PTH POWER OF THE FIRST N INTEGERS

Consider the series formed by the sum of the p th power of each of the first N integers

$$S[p, N] = \sum_{n=1}^N n^p = 1 + 2^p + 3^p + \dots + N^p$$

and ask “what is the value of this sum?”. Starting with the simplest case of $p=1$, one sees at once on regrouping that-

$$S[1, N] = (1 + N) + (2 + N - 1) + (3 + N - 2) + \dots$$

so that the sum must be the quantity $(1+N)$ repeated $N/2$ times. Hence

$$S[1, N] = N(N+1)/2$$

One of the stories often told in connection with this last sum is that involving young K.Gauss in elementary school. One day his teacher wanted to have the class remain quiet for a half hour or so and did this by giving them the task of coming up with the sum of first hundred integers. To the teachers surprise Karl responded almost immediately with the correct answer of 5050, having worked out the $S[1, N]$ formula in his head and thus avoiding the laborious effort of actually adding each integer in sequence $1+2+3+4+\dots+100$ as the rest of the class was doing.

On examining $S[1, N]$ one notices that the sum is represented by a quadratic in N . This suggests that the sum $S[2, N]$ is probably given by a cubic in N and $S[p, N]$ by a $(p+1)$ order polynomial in N .

Lets apply this assumption by trying-

$$S[2, N] = AN^3 + BN^2 + CN \text{ with } 1 = A + B + C, \quad 5 = 8A + 4B + 2C, \quad 14 = 27A + 9B + 3C$$

The last three algebraic equations for the constants A , B and C are obtained by evaluating $S[2, N]$ for $N=1, 2$, and 3 . A simple evaluation shows that these coefficients have the values $1/3, 1/2$, and $1/6$, respectively. Thus the resultant equation when $p=2$ becomes-

$$S[2, N] = N(2N+1)(N+1)/6$$

A simple test that this is the correct result follows by noting that $S[2, 3] = 1+4+9 = 3(7)(4)/6 = 14$. Also, as expected, the coefficient for the highest power of N in our expression is what is gotten

when integrating N^2 .

Next looking at $p=3$ we expect a sum to be of the form $S[3,N]=AN^4+BN^3+CN^2+DN$ with the constants A,B,C,D to be determined by solving the matrix equation $MF=G$ where -

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 \\ 81 & 27 & 9 & 3 \\ 256 & 64 & 16 & 4 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 9 \\ 36 \\ 100 \end{bmatrix}$$

and F is the coefficient matrix whose solution is-

$$F = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \\ 0 \end{bmatrix}$$

One thus finds the summation formula-

$$S[3,N] = \sum_{n=1}^N n^3 = N^2(N+1)^2/4$$

Following the above procedure for $p=4$ and $p=5$, leads to-

$$S[4,N] = N(6N^4 + 15N^3 + 10N^2 - 1)/30$$

and

$$S[5,N] = N^2(2N^4 + 6N^3 + 5N^2 - 1)/12$$

The coefficients found for any positive integer value of p can also be expressed in terms of the coefficients of Bernoulli polynomials of order $p+1$ (see **Numerical Calculations in Engineering** by **J.Tuma**). It is also a simple matter, using the above results, to show that odd integers sum to

$$\sum_{n=0}^N (2n+1) = (N+1)^2$$