Question-1

(Problem 2.3 of Arora’s Introduction to Optimum Design): Design a beer mug, shown in fig, to hold as much beer as possible. The height and radius of the mug should be not more than 20 cm. The mug must be at least 5cm in radius. The surface area of the sides must not be greater than 900cm² (ignore the area of the bottom of the mug and ignore the mug handle—see fig). Formulate the optimal design problem.

![Diagram of a beer mug](image)

**Problem formulation:**

**Design variables:** $X = (R, H)$

**Objective function:**
- maximize volume, $\pi R^2 H$
- Minimize, $f(X) = - \pi R^2 H$

**Subjected to:**
1) $H \leq 20$ OR $g_1(X) = H - 20 \leq 0$
2) $5 \leq R \leq 20$ OR $g_2(X) = R - 20 \leq 0$
3) $2\pi RH \leq 900$ OR $g_3(X) = 2\pi R H - 900 \leq 0$
4) $H \geq 0$

**Graphical solution:**
Graphical solution is shown on the next page.
Objective function and constraints are plotted in the design domain.
Feasible domain is contained by constraint boundaries as marked in the plot.
Optimum point is marked by black dot.
Constraint $g_2$ and $g_4$ are active at the optimum point.
Minimum $f(X)$ is found to be $-9000$ cm³ at $R=20$ cm and $H = 7.16$ cm
At Optimum point:
f(X) = -9000 cm³
R = 20 cm
H = 7.16 cm

Maximum Volume = 9000 cm³ at R=20 cm and H = 7.16 cm
Question-2

(Problem 2.16 of Arora's Introduction to Optimum Design): Design of a two-bar truss. Design a symmetric two bar truss (both members have the same cross section) shown in figure to support a load W. The truss consists of two steel tubes pinned together at one end and supported on the ground at the other. The span of the truss is fixed at s. formulate the minimum mass truss design problem using height and the cross-sectional dimension as design variable. The design should satisfy the following constraint.

1. Because of the space limitation, the height of the truss must not exceed b_1, and must not be less than b_2.
2. The ratio of the mean diameter to thickness of the tube must not exceed b_3.
3. The compressive stress in the tubes must not exceed the allowable stress \( \sigma_a \) for steel.
4. The height, diameter, and thickness must be chosen to safeguard against member buckling.

Use the following data: \( W=10kN \); span \( s=2m \); \( b_1 = 5m \); \( b_2 = 2m \); \( b_3 = 90 \); allowable stress, \( \sigma_a = 250 \) MPa; modulus of elasticity, \( E = 210 \) GPa; mass density, \( \rho = 7850 \) kg/m\(^3\); factor of safety against buckling, \( FS = 2 \); \( 0.1 \leq D \leq 2 \) (m); and \( 0.01 \leq t \leq 0.1 \) (m).

**Problem formulation:**

\[
\begin{align*}
F_1 \text{ and } F_2 & \text{ are the forces in the two truss bars.} \\
\text{Let } L \text{ be the length of each bar and } s \text{ be the distance between two end points of bars (as shown in fig).} \\
\text{Let } t \text{ be the thickness of cross section.} \\
\text{Due to the symmetry, } & \ F_1 = F_2 = F \\
\text{Also, } & \ \Sigma F_y = 0 \Rightarrow -2F\cos(\alpha) + W = 0 \\
& \Rightarrow F = W/2\cos(\alpha) \\
\text{Assuming } t << D, \text{ cross-section area of truss bar, } A = \pi Dt \\
\text{Length of bar } & \ L = \sqrt{H^2 + (.5s)^2} \\
\text{Stress in truss member, } & \ \sigma_c = F/A = W/[2\cos(\alpha) \pi Dt]
\end{align*}
\]
For pinned-pinned column, critical buckling load, $P_{cr} = \frac{\pi^2 E I}{L^2}$
Where, moment of inertia, $I = \pi D^4 t/8$ and $E$ is young's modulus

**Design variables:** $X = (H,D,t)$

**Objective function:** Minimize the mass of truss
Minimize, $f(X) = 2\rho A L$

**Subjected to:**
1) $D/t \leq b_3$ OR $g_1(X) = D - b_3 t \leq 0$
2) $\sigma_c \leq \sigma_a$ OR $g_2(X) = \sigma_c - \sigma_a \leq 0$
3) $F \leq P_{cr}/2$ OR $g_3(X) = F - P_{cr}/2 \leq 0$
4) $2 \leq H \leq 5$
5) $0.1 \leq D \leq 2$
6) $0.01 \leq t \leq 0.1$

**Graphical solution:**
Since we have 3 design variables, to solve the problem graphically, we take 2 variables at a time.

Here we choose to take $D$ and $t$ as 2 design variables in design domain while fixing value of $H$ at different levels (min, mean, max) and checking its effect on objective function and if any of the constraints are violated.

1) Minimum $H = 2$ m
2) Mean $H = 3.5$ m

3) Maximum $H = 5$ m
As we can see, constraint $g_1$ is active in the design domain, whereas constraints $g_2$ and $g_3$ are not active in any design domain tested here. (for three values of $H$)

**Working in minimum $H$ design domain, $H = 2$ m**

Optimum point: $H = 2$ m, $D = 0.1$ m and $t = 0.01$ m

Minimum mass of truss, $f(X) = 110.29$ kg
**Question-3**

(Problem 2.17 of Arora's Introduction to Optimum Design): A beam of rectangular cross section as shown in figure is subjected to maximum bending moment of M and maximum sheer of V. The allowable bending and shearing stresses are \( \sigma_a \) and \( \tau_a \) respectively. The bending stress in the beam is calculated as \( \sigma = 6M/bd^2 \) and average shear stress in the beam is calculated as \( \tau = 3V/2bd \). Where d is the depth and b is the width of the beam. It is also desired that the depth of the beam shall not exceed twice its width. Formulate the design problem for minimum cross sectional area using the following data: \( M = 140 \text{kN.m}, V=24 \text{kN} \) \( \sigma_a = 165 \text{MPa} \) \( \tau_a = 50 \text{MPa} \)

**Problem formulation:**
Bending stress, \( \sigma = 6M/bd^2 \)
Shear stress, \( \tau = 3V/2bd \)

**Design variables:** \( X = (d, b) \)

**Objective function:** minimize the cross sectional area
Minimize, \( f(X) = bd \)

**Subjected to:**
1) \( d \leq 2b \) OR \( g_1(X) = d - 2b \leq 0 \)
2) \( \sigma \leq \sigma_a \) OR \( g_2(X) = \sigma - \sigma_a \leq 0 \)
3) \( \tau \leq \tau_a \) OR \( g_3(X) = \tau - \tau_a \leq 0 \)

4) \( b, d \geq 0 \)

**Graphical solution:**
For graphical solution, following design domain is shown: \( 0 \leq b \leq 300, \ 0 \leq d \leq 300 \) (mm) Contours of objective function and constraints are plotted in the design domain. Optimum point is marked by black dot. Constraint \( g_1 \) and \( g_2 \) are active at the optimum point.
Minimum \( f(X) \), cross sectional area of beam, is found to be \( = 23500 \text{ mm}^2 \)

Optimum design: \( b = 108 \text{ mm}, d = 217 \text{ mm} \) (constraints \( g_1 \) and \( g_2 \) are active)
Question-4

(Problem 2.17 of Arora’s Introduction to Optimum Design): Design a hollow circular beam shown in figure for two conditions when P = 50KN, the axial stress $\sigma$ should be less than $\sigma_a$, and when P = 0, deflection $\delta$ due to self weight should satisfy $\delta \leq 0.001 l$. The limits for dimension are $t = 0.10$ to 1.0 cm, $R = 2.0$ to 20.0 cm. Formulate the minimum weight design problem and transcribe it into the standard form. Use the following the data: $\delta = 5wl^4/384EI$; $w = \text{self weight force/length (N/m)}$; $\sigma_a = 250\text{MPa}$; modulus of elasticity, $E = 210\text{GPa}$; mass density; $\rho = 7800\text{kg/m}^3$; $\sigma = P/A$; gradational constant, $g = 9.9\text{m/s}^2$; moment of inertia, $I = \pi R^4 t$ (m$^4$).

**Problem formulation:**

Part 1: $P = 50KN$

Part 2: $P = 0$

Beam cross section area, $A = 2\pi Rt$

Axial stress, $\sigma = P/A$

$w = \text{self weight force/length} = \rho g 2\pi Rt/l = 7800*9.9*2\pi Rt$

Moment of inertia, $I = \pi R^4 t$

Displacement under self weight: $\delta = 5wl^4/384EI$

**Design variables:** $X = (R, t)$

**Objective function:** minimization of weight $= \rho g 2\pi Rt$

Minimize, $f(X) = 6\pi \rho g R t$

**Subjected to:**

1) $\sigma \leq \sigma_a$, OR $g_1(X) = (\sigma / \sigma_a) - 1 \leq 0$

2) $\delta - 0.001 l \leq 0$, OR $g_2(X) = (\delta / 0.001 l) - 1 \leq 0$

NOTE: IT IS GOOD TO WORK WITH NON-DIMENSIONAL NORMALIZED CONSTRAINTS

3) $0.1 \leq t \leq 1$

4) $2 \leq R \leq 20$

**Graphical solution:**

Figure on the next page shows contour plots of the objective function with tube thickness on x-axis and mean radius of tube cross section on the y-axis.

Constraints are plotted on the graph and the feasible domain is as marked.
\[
g_1 = 0 \\
g_2 = 0
\]

Feasible region

Optimum solution line

R ≥ 0.02
One objective function contour line follows constraint \( g_1 \), thus we have an optimum solution line as indicated by the solid black line. All the points on this line are optimum and minimize the beam weight.

**Minimum \( f(X) \), beam weight = 46.4 N**

Sample optimum designs:

1. \( R = 0.02 \, \text{m}, \, t = 0.0016 \, \text{m} \) (constraints \( g_1 \) and \( R \geq 0.02 \) are active)
2. \( R = 0.032 \, \text{m}, \, t = 0.001 \, \text{m} \) (constraints \( g_1 \) and \( t \geq 0.001 \) are active)
**Question-5**

Consider a laminate made from graphite/epoxy with $E_1=128$ GPa, $E_2=13$ GPa, $G_{12} = 6.4$ GPa, and $v_{12} = 0.3$. Formulate the design problem of obtaining an 8-ply symmetric balanced laminate with maximum $E_x$, such that $G_{xy}$ is at least 25 GPa, and $v_{xy}$ is not bigger than 1.

**Problem formulation:**

$E_1 = 128$ GPa, $E_2 = 13$ GPa, $G_{12} = 6.4$ GPa and $v_{12} = 0.3$

$v_{21} = E_2^*/v_{12}/E_1 = 0.030$

Also,

$Q_{11} = E_1/(1 - v_{12}^* v_{21}) = 128*10^9/(1-0.009) = 129.16*10^9$

$Q_{22} = 13.11*10^9$

$Q_{12} = 3.93*10^9$

$Q_{66} = 6.4*10^9$

Also we can find the material invariants by:

$U_1 = (3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66})/8 = 57.53*10^9$

$U_2 = (Q_{11} - Q_{22})/2 = 58.02*10^9$

$U_3 = (Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66})/8 = 13.6*10^9$

$U_4 = (Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66})/8 = 17.55*10^9$

$U_5 = (Q_{11} + Q_{22} - 2Q_{12} + 4Q_{66})/8 = 20*10^9$

Thickness of individual ply is assumed to be constant = $t$

For balanced laminate we will have the following stacking sequence, $(\theta_1/ – \theta_1 / \theta_2/ – \theta_2)$,

$V_{1A} = 4t [\cos 2 \theta_1 + \cos 2 \theta_2 ]$

$V_{3A} = 4t [\cos 4 \theta_1 + \cos 4 \theta_2 ]$

And $V_{1A}' = V_{1A}/h = V_{2A}/8t = [\cos 2 \theta_1 + \cos 2 \theta_2 ]/2$

And $V_{3A}' = V_{3A}/h = V_{3A}/8t = [\cos 4 \theta_1 + \cos 4 \theta_2 ]/2$

Then

$A_{11}/h = [U_1 + U_2 + U_3 + V_{1A}']$

$A_{22}/h = [U_1 - U_2 + U_3 + V_{3A}']$

$A_{12}/h = [U_4 + U_6 + V_{3A}]$

$A_{66}/h = [U_5 - U_3 + V_{3A}]$

**Design variables:**

$X = (\theta_1, \theta_2)$

**Objective function:**

Maximize $E_x = [(A_{11}/h)*(A_{22}/h) – (A_{12}/h)^2] / (A_{22}/h)$ OR Minimize, $f(x) = - [(A_{11}/h)*(A_{22}/h) – (A_{12}/h)^2] / (A_{22}/h)$

**SubJECTED to:**

1) $G_{xy} = (A_{66}/h) \geq 25*10^9$ OR $g_1(X) = 1- [(A_{66}/h)/ 25*10^9] \leq 0$

2) $v_{xy} = A_{12}/A_{22} \leq 1$ OR $g_2(X) = (A_{12}/A_{22}) - 1 \leq 0$

3) $0 \leq \theta_1 \leq 90$

4) $0 \leq \theta_2 \leq 90$
Graphical solution:
Graph below shows contours of objective function with constraints $g_1=0$ and $g_2 = 0$ marked. Feasible domain is as marked on the plot.

Design (1) and (2) both are simultaneously optimum with minimum $f(X) = -60$ GPa. Both the constraints $g_1$ and $g_2$ are active.

Maximum $E_x = 60$ GPa
Layup sequence $=(\theta_1/ \theta_1/ \theta_2/ \theta_2)$.
With,  Design (1) $\theta_1 = 21.2^\circ$, $\theta_2 = 36.4^\circ$
Design (2) $\theta_1 = 36.4^\circ$, $\theta_2 = 21.2^\circ$
Annexure – 1
Matlab code used for problems

**Problem 1**

clear all
close all

% Design variables
[xR,yH] = meshgrid(2:0.5:22,-2:0.5:22);

% Objective function
f = -pi()*xR.^2.*yH;

% Constraints
g1 = yH - 20;
g2 = 5 - xR;
g3 = xR - 20;
g4 = 2*pi*xR.*yH - 900;
g5 = -yH;

% plotting
fv = [0:-3000:-30000];
cs = contour(xR,yH,f,fv); clabel(cs)
hold on
cs1 = contour(xR,yH,g1,[0 0],'r'); clabel(cs1)
cs2 = contour(xR,yH,g2,[0 0],'b'); clabel(cs2)
cs3 = contour(xR,yH,g3,[0 0],'b'); clabel(cs3)
cs4 = contour(xR,yH,g4,[0 0],'p'); clabel(cs4)
cs5 = contour(xR,yH,g5,[0 0],'r'); clabel(cs5)
xlabel('R (Radius, cm)')
ylabel('H (Height, cm)')
hold off

**Problem 2**

clear all
close all

% Design variables
xt,yD,zH] = ndgrid(0.01:0.001:0.1,0.1:0.01:2,2:0.1:5);

% Objective function
f = 2*7850*pi()*yD.*xt.*sqrt(zH.^2+1);

% Constraints
W = 10*1000;
E = 210*le9;
g1 = yD - 90.*xt;
g2 = W/(2*(zH./sqrt(zH.^2+1)*pi().*yD.*xt))-250*le6;
g3 = (W*sqrt(zH.^2+1)./zH)-((pi()^3*E*yD.^3.*xt)./(8*(zH.^2+1)));
% plotting
% 3) Try D, t at various H
i = 1; % H = 2 m
i = 4; % H = 3.5 m
i = 7; % H = 5 m
fv = [500:1000:1e4];
cs = contour(xt(:,:,i),yD(:,:,i),f(:,:,i),fv); clabel(cs)
hold on
cs1 = contour(xt(:,:,i),yD(:,:,i),g1(:,:,i),[0,0],'r'); clabel(cs1)
% cs2 = contour(xt(:,:,i),yD(:,:,i),g2(:,:,i),[-2.498e8,-2.498e8],'b'); clabel(cs2)
% cs3 = contour(xt(:,:,i),yD(:,:,i),g3(:,:,i),[-0.5e7,-0.5e7],'c'); clabel(cs3)
% cs3 = contour(xt(:,:,i),yD(:,:,i),g3(:,:,i),[0,0],'c'); clabel(cs3)
xlabel('t (Tube thickness, m)')
ylabel('D (Tube diameter, m)')
hold off

Problem 3

clear all
close all

% Design variables
[xb,yd] = meshgrid(0:1:300,0:1:300);

% Objective function
f = xb.*yd;

% Constraints
g1 = yd - 2*xb;
g2 = (6*140*1e6./(xb.*yd.^2)) - 165;
g3 = (3*24*1e3./(2*xb.*yd)) - 50;

% plotting
fv = [0:10000:60000,23500];
cs = contour(xb,yd,f,fv); clabel(cs)
hold on
cs1 = contour(xb,yd,g1,[0,0],'r'); clabel(cs1)
cs2 = contour(xb,yd,g2,[0,0],'b'); clabel(cs2)
cs3 = contour(xb,yd,g3,[0,0],'g'); clabel(cs3)
xlabel('b (Beam width, mm)')
ylabel('d (Beam depth, mm)')
hold off
Problem 4

clear all
close all

% Design variables
[xt,yR] = meshgrid(0.1/100:0.005/100:1/100,0/100:0.05/100:20/100);

% Zoom plot
% [xt,yR] = meshgrid(0.1/100:0.005/100:0.22/100,0:0.05/100:5/100);

% Objective function
f = 6*pi()*7800*9.9*yR.*xt;

% Constraints
g1 = (1./(10000*pi()*yR.*xt)) - 1;
g2 = ((5*7800*9.9*2*3^3)./(0.001*384*210*1e9*yR.^2)) - 1;
g3 = 2/100 - yR;

% Plotting
fv = [0:100:1000];
cs = contour(xt,yR,f,fv); clabel(cs)

% Zoom plot
% fv = [10:5:80];
% cs = contour(xt,yR,f,fv); clabel(cs)

hold on
cs1 = contour(xt,yR,g1,[0,0],'r'); clabel(cs1)
cs2 = contour(xt,yR,g2,[0,0],'b'); clabel(cs2)
cs3 = contour(xt,yR,g3,[0,0],'g'); clabel(cs3)
xlabel('t (m)')
ylabel('R (m)')
hold off

Problem 5

clear all
close all

% Design variables
incr = 0.1;
tstart = 0;
tend = 90;
[xthe1,ythe2] = meshgrid(tstart:incr:tend,tstart:incr:tend);

E1 = 128;
E2 = 13;
G12 = 6.4;
v12 = 0.3;
v21 = E2* v12/E1;
Q11 = E1/(1-v12*v21);
Q22 = E2/(1-v12*v21);
Q12 = v12*Q22;
Q66 = G12;

% material invariants
U1 = (3*Q11 + 3*Q22 + 2*Q12 + 4*Q66)/8;
U2 = (Q11 - Q22)/2;
U3 = (Q11 + Q22 - 2*Q12 - 4*Q66)/8;
U4 = (Q11 + Q22 + 6*Q12 - 4*Q66)/8;
U5 = (Q11 + Q22 - 2*Q12 + 4*Q66)/8;

Vs1A = 0.5*(cos(2*xthe1*pi()/180)+cos(2*ythe2*pi()/180));
Vs3A = 0.5*(cos(4*xthe1*pi()/180)+cos(4*ythe2*pi()/180));

A11h = U1+U2*Vs1A+U3*Vs3A;
A22h = U1-U2*Vs1A+U3*Vs3A;
A12h = U4-U3*Vs3A;
A66h = U5-U3*Vs3A;

% Objective function
f = -((A11h.*A22h)-A12h.^2)./(A22h);

% Constraints
g1 = 1 - (A66h./25);
g2 = (A12h./A22h) - 1;

% plotting
cs = contour(xthe1,ythe2,f); clabel(cs)
hold on
cs1 = contour(xthe1,ythe2,g1,[0,0],'r'); clabel(cs1)
cs2 = contour(xthe1,ythe2,g2,[0,0],'b'); clabel(cs2)
xlabel('theta 1 (deg)')
ylabel('theta 2 (deg)')
hold off