First order reliability method (FORM)

- Limit state \( g(X) \). Failure when \( g<0 \).
- Linear approximation of limit state together with assumption that random variables are normal.
- Then limit state is also normal variable.
- Reliability index is the distance of the mean of \( g \) from zero measured in standard deviations.
Approximation about mean

- Predecessor of FORM called first-order second-moment method (FOSM)

\[ \tilde{g}(X) = g(\mu_x) + \nabla g(\mu_x)^T(X - \mu_x) \]

Then, easy to show

\[ \mu_{\tilde{g}} = g(\mu_x) \quad \sigma_{\tilde{g}} = \left[ \sum_{i=1}^{n} \left( \frac{\partial g}{\partial x_i} \right)^2 \sigma_{x_i}^2 \right]^{1/2} \]

The reliability index and probability of failure are

\[ \beta = \frac{\mu_{\tilde{g}}}{\sigma_{\tilde{g}}} \]

\[ P_f = \Phi(-\beta) \quad \Phi \text{ is the normal CDF} \]
Beam under central load example

- Probability of exceeding plastic moment capacity
  \[ g(P, L, W, T) = WT - PL / 4 \]

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>L</th>
<th>W (plastic section modulus)</th>
<th>T (yield stress)</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>10kN</td>
<td>8m</td>
<td>0.0001m^3</td>
<td>600,000 kN/m^2</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>2kN</td>
<td>0.1m</td>
<td>0.00002m^3</td>
<td>100,000kN/m^2</td>
</tr>
</tbody>
</table>
Reliability index for example

• Using the linear approximation get

\[ \mu_g = g(\mu_x) = 0.0001 \times 600,000 - 10 \times 8 / 4 = 40 \text{kNm} \]

\[ \frac{\partial g}{\partial T} = W = 0.0001 \quad \frac{\partial g}{\partial W} = T = 600,000 \]

\[ \frac{\partial g}{\partial P} = -L / 4 = -2 \quad \frac{\partial g}{\partial L} = -P / 4 = -2.5 \]

\[ \sigma_g = \left[ \sum_{i=1}^{n} \left( \frac{\partial g}{\partial x_i} \right)^2 \sigma_{x_i}^2 \right]^{1/2} = \sqrt{0.0001 \times 100,000^2 + (600,000 \times 0.000002)^2 + (-2 \times 2)^2 + (-2.5 \times 0.1)^2} = 16.13 \text{kNm} \]

\[ \beta = \frac{\mu_g}{\sigma_g} = 2.48 \]

• Chapter 4 of CGC shows that if we change to \( g = T - 0.25PL/W \) we get 3.48 instead (exact is 2.46)
Most probable point (MPP)

• The error due to the linear approximation is exacerbated due to the fact that the expansion may be about a point that is far from the failure region (due to the safety margin).
• Hasofer and Lind suggested remedying this problem by finding the most probable point and linearizing about it.
• The joint distribution of all the random variables assigns a probability density to every point in the random space. The point with the highest density on the line $g=0$ is the MPP.
\begin{verbatim}
r = randn(1000,1)*1.25+10;
c = randn(1000,1)*1.5+13;
f = @(x) x;
fplot(f,[5,20])
hold on
plot(r,c,'ro')
xlabel('r')
ylabel('c')
\end{verbatim}
Recipe for finding MPP with independent normal variables

• Transform into standard normal variables (zero mean and unity standard deviation)
  \[ u_i = \frac{x_i - \mu_{x_i}}{\sigma_{x_i}} \]

• Find the point on \( g=0 \) of minimum distance to origin. The point will be the MPP and the distance to the origin will be the reliability index based on linear approximation there.

\[ \beta = \min_{U \in g(U) = 0} \left( U^T U \right)^{1/2} \]
Visual

Figure 4.2. Mapping of Failure Surface from X-space to U-space
General case

• If random variables are normal but correlated, a linear transformation will transform them to independent variables.

• If random variables are not normal, can be transformed to normal with similar probability of failure. See Section 4.1.5 of CGC.
Project application

• The two thickness variables are normal and independent, so transforming to U’s is easy.

• To find the most probable point can use optimization in either Matlab or EXCEL