Three-phase plane composites of minimal elastic stress energy

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In this work we establish tight lower bound for effective elastic stress energy of two-dimensional three-phase anisotropic composites. It is assumed that the materials are mixed with fixed volume fractions and that one of the phases is degenerated to void, i.e. the effective material is porous. The bound is derived using a combination of the translation method and additional inequalities on the stress fields in materials. In the context of elasticity, similar technique was used by Cherkaev (Mech. Mater. 41(4), 411-433, 2009) for isotropic multiphase composites; in the present work we elaborate the case of full anisotropy. Our goal is to expand the Hashin–Shtrikman and classical translation bounds to multiphase structures.

Motivation of our work stems from the fact that the vast majority of available results deals with two-material composites. Meanwhile, numerous applications call for optimal design of multi-material mixtures, or even porous composites from two elastic materials and void, especially applications that utilize multi-physics, i.e. elastic and electromagnetic properties, and those that deal with structures best adapted to variable environment such as natural morphologies perfected by evolution.

Optimal microstructures of two-phase and multiphase composites are drastically different. In contrast with the steady and intuitively expected topology of two-material optimal mixture (a strong material always surrounds weak inclusions), optimal multi-material structures show the large variety of patterns and the optimal topology depends on volume fractions of constituent phases. Optimal structure may contain an enveloping layer but it also has 'hubs' of a material with intermediate stiffness connected by 'pathways' (laminates of strong and weak materials) and other configurations that reveal a geometrical essence of optimality. Geometries of multi-material optimal structures are not unique, pieces of the same material may occur in different places of an optimal structure and they may correspond to different stress fields inside them. Consequently, the method for finding optimal multiphase geometries differs from the methods developed for optimal two-material structures.

The existing techniques for the bound such as Hashin-Shtrikman method, translation method, or analytic method of Bergman-Milton produced a number of results for two-material mixtures in the last 25 years but they do not provide exact results for multi-material problems. This poses a mathematical challenge which is addressed in this research.

At the present stage of research, it is shown that the lower bound on the effective stress energy is a multifaceted surface. The structure of optimal fields in each material of the composite is analyzed and described in the high-porosity regime where the obtained energy estimate is exact and it is realized on microstructures in the form of high-rank laminates whose optimal geometrical parameters are explicitly found. Corresponding formulae solving the G-closure problem of a three-phase composite are also derived. In low-porosity regime the bound and the conjuncted optimal microstructures are still slightly different. Our guess is that there is another, yet unaccounted, inequality which becomes active in the latter case. Full discussion of this issue is postponed to the subsequent stage of the research.