10th World Congress on Structural and Multidisciplinary Optimization
May 19 -24, 2013, Orlando, Florida, USA

On the optimal topologies considering uncertain load positions

János Lógó[1], Erika Pintér[2], Anna Vásárhelyi[3]

Department of Structural Mechanics, Budapest University of Technology and Economics
H-1111 Budapest, Műegyetem rkp. 3, Hungary
[1] logo@ep-mech.me.bme.hu, [2] epinter@mail.bme.hu, [3] anna.vasarhelyi@freemail.hu

1. Abstract
A new numerical method is presented for the continuum type topology optimization problems in the case of uncertain loading positions. The optimization problem is a volume minimization one subjected to probabilistic compliance constraints. In addition to the optimization procedure a parametric study is presented to investigate the layout theory. It is proved that not only statically determinate but statically undeterminate structure can be the optimal layout.
2. Keywords: Optimal layout, topology optimization, probabilistic loading, uncertain point of application, multiply load case.

3. Introduction
The more than century old topology optimization has a relatively young new research direction, namely considering probabilistic data in the topology design. Uncertainty is typically limited to the loading, although recent works have considered extensions to support conditions, and material properties, etc.. [6-8].
In this paper the loading positions are taken as stochastic variables and all the other data are deterministic. The linearly elastic discretized structure is modelled mechanically by plane stress disk elements. To make the optimization method robust an equivalent deterministic problem is derived [1-3,5]. The elaborated technique can be described as follows: it is assumed that the load positions are given by either their distribution function, mean value and covariance matrix or by the simple values of the probability of the occurrence of a force at a certain location. The first case, where the statistical information (distribution functions, mean values and variations) are given, is always finished by a simple calculation which results in the probability values of the occurrence of a force at a certain location, that is practically in the second case. Hence each load is considered in an extended loading domain. Since the loading positions are not known precisely, an equivalent loading system should also be created around the expected location of each force to perform a "simulation". According to the original distribution assumption, mean value and variation of the point applications, an extended force system is set up for each possible loading domain with the original magnitude of the force and given (or calculated) probability values. Each load is independent and acts as an independent load case in the original loading domain. Applying these forces at these "base" points as loads the stochastic design problem becomes a deterministic one. By the use of the elements of this force system one by one, the displacement vectors can be calculated from the usual linear equilibrium equations with several load cases. Since the material is linearly elastic, isotropic and homogenious, the additive properties of the displacements and the *reciprocity theorem* can be applied. Using these vectors and the assigned probability values the expected displacement and its variation can be calculated. By the use of these data the original compliance value, which is probabilistic due to the position uncertainties, can be substituted with a deterministic one applying the Kataoka theorem [4]. Using this compliance formulation either a min-max objective function is formed, which is composed by the expected compliance and a certain type of variance of the compliance due to the independent load cases or a volume minimization problem is created what is subjected to several compliance constraints. Here the later one is used as base problem. At first case the constraints are the volume limitation and the side values of the design variables. In the case of Gaussian distribution of the displacement field the unconstraint problem objective function is simplified to a function minimization also due to the Kataoka theorem [4]. An extended SIMP type algorithm is elaborated for the solution method. To validate the model a deterministic minimum compliance truss design is performed analytically and numerically.

Several numerical examples are presented.

4. Notes on the layout theory

The minimum weight design as an objective was a rather popular topic during “golden ages” of the optimization (e.g. during the 50-s to 70-s of the last century). The authors investigated whether the statically determinate or undeterminate structure gives the optimal layout [8-10] with minimum weight. It is known in engineering design that a statically determinate structure is not sensitive for kinematic loading, but any change in static loading may produce an unexpected collapse. In this way the statically undeterminate structures can be more safe for unexpected load cases (see: the structure of the bones). This question is rather difficult whenever the loading uncertainity is investigated. In this case the load can be considered as a quantity given in an interval.

In this paper at first it is proved trough two simple examples that one can construct several (infinite number) alternative statically undeterminate structures having the same volume and compliance value if a statically determinate structure exists.

4.1. Simple examples for equivalent determinate versus undeterminate structures

The first example is a 3-bar truss -as a base structure- with a vertical force at the top (Fig.1). The material is linearly elastic (for sake of simplicity the Young’s modulus E=1) and the vertical load is 100. The members are supported by hinges at the bottom. The total compliance is calculated as follow:

$$C=\sum_{i=1}^{n}\frac{F_i^2 L_i}{EA_i} \quad (1)$$

where F_i = the elastically calculated force in member i, L_i = length of member i, A_i = cross-sectional area of member i.

One can see in Table 1. that mechanically same (same volume and same compliance) structures can be composed by simple modification of the number of the members (doubled the bars and mechanically equivalently decrease the cross-sectional areas of the members). Here a 6 and 12-bar structures were calculated as examples.

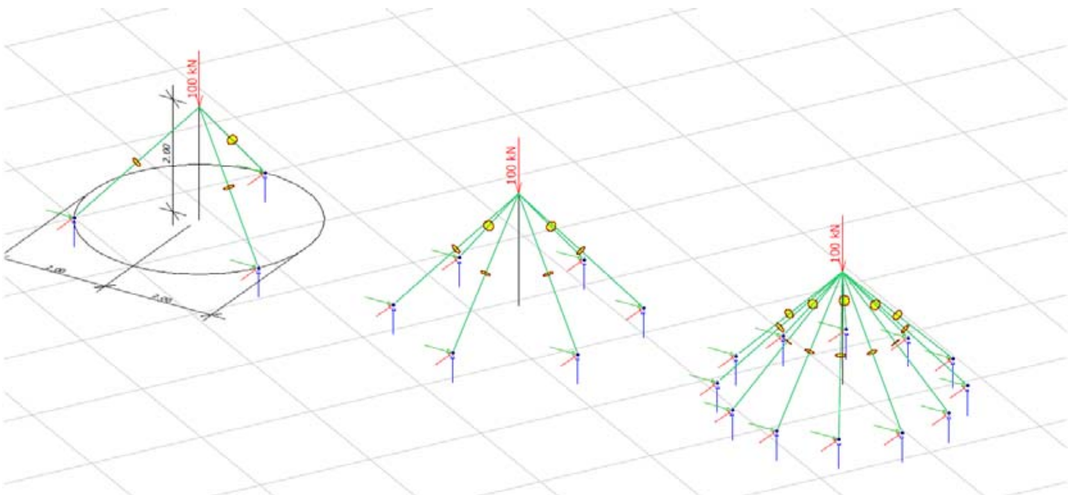

Figure 1: Alternative truss laouts in the case of a vertical point load at the top

Table 1: Comparative values of the optimal 3, 6 and 12-bar structures in the case of a vertical point load at the top

length	**pc**	**section (cm2)**	**volume**	**Normal force**	**stress**	**top displacement**	**Compliance (external Pot. energ.)**	**Compliance of the bars (strain energy)**
2,82842	3	1,57	1332,189	47,14	-3,0025	0,0572	5,72	5,719083764
2,82842	6	0,785	1332,189	23,57	-3,0025	0,0572	5,72	5,719083764
2,82842	12	0,3925	1332,189	11,785	-3,0025	0,0572	5,72	5,719083764
σ=2100								
Young's	210000N/mm2							

A very similar example can be calculated if the top vertical force (100) is modified and a horizontal force (57,74) is added (see Fig 2.) All the other data and the way of the calculations are the same. The results of the calculations can follow from the details of Table 2.

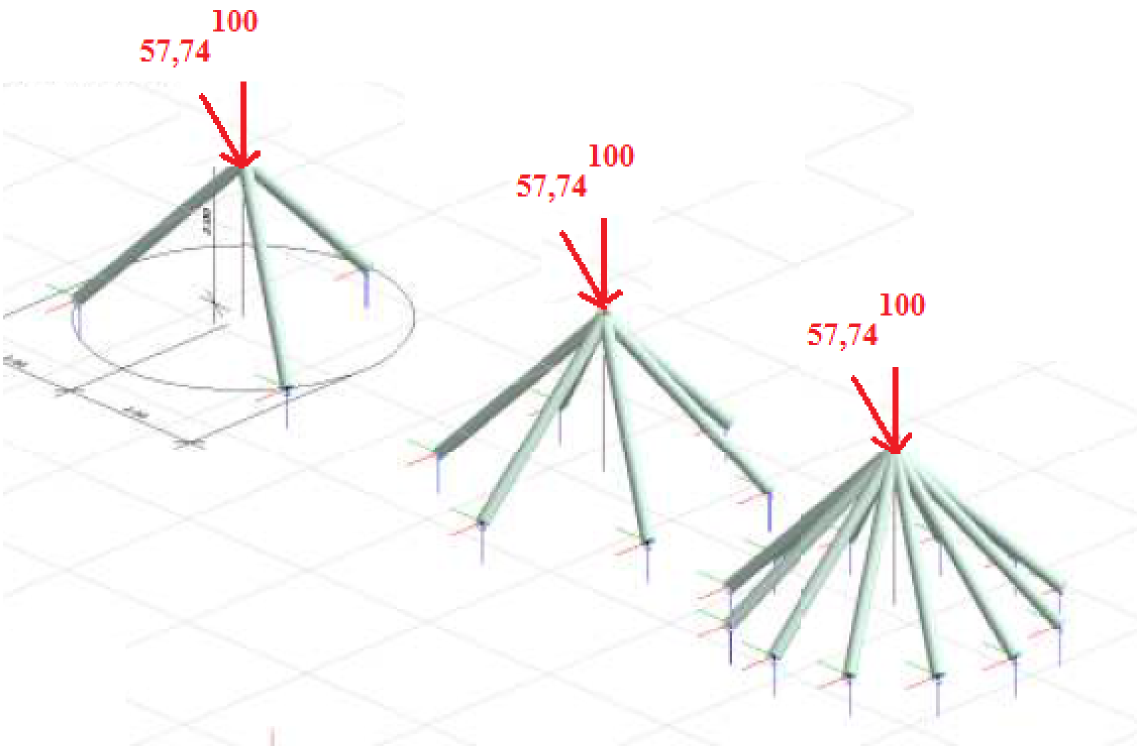

Figure 2: Alternative truss problems in the case of two point loads at the top

Table 2: Comparative values of the optimal 3, 6 and 12-bar structures in the case of two point loads at the top

length	pc	section (cm2)	volume	Normal force	stress	top displacement	Compliance (external Pot. energ.)	Compliance of the bars (strain energy)
2,828427	1	1,57	444,0631	101,58	100	0,0572	5,72	8,852022
2,828427	2	1,57	888,1261	19,92	57,74	0,0572	3,812957	0,680824
		total	**1332,189**			0,033	**9,532957**	**9,532845**
						6 bar truss		
2,828427	1	0,785	222,0315	3,65	100	0,0572	5,72	0,022858
2,828427	2	0,785	444,0631	9,96	57,74	0,0572	3,812957	0,340412
2,828427	2	0,785	444,0631	37,18		0,033		4,743564
2,828427	1	0,785	222,0315	50,79				4,426011
		total	**1332,189**				**9,532957**	**9,532845**
						12 bar truss		
2,828427	1	0,3925	111,0158	1,82	100	0,0572	5,72	0,011367
2,828427	2	0,3925	222,0315	0	57,74	0,0572	3,812957	0
2,828427	2	0,3925	222,0315	4,98		0,033		0,170206
2,828427	2	0,3925	222,0315	11,79				0,95399
2,828427	2	0,3925	222,0315	18,59				2,371782
2,828427	2	0,3925	222,0315	23,57				3,812723
2,828427	1	0,3925	111,0158	25,39				2,212134
		total	**1332,189**				**9,532957**	**9,532201**

As a conclusion of the calculation above one can state if a statically determinate structure exists as a solution of a deterministic problem with a single load case, several (infinite number) statically equivalent undeterminate structures can be derived with the same weight and the compliance.

4.2. Minimum volume design of structures according to the optimal layout theory
In the case of probabilistic loading the magnitude, the line of action, the direction and the point of application of the load can be uncertain. Here trough a simple example it is proved that not only one type of layout can be optimal. There will be singular layout solutions for certain case or the optimal layout can be changed if the magnitude of the horizontal load is uncertain.

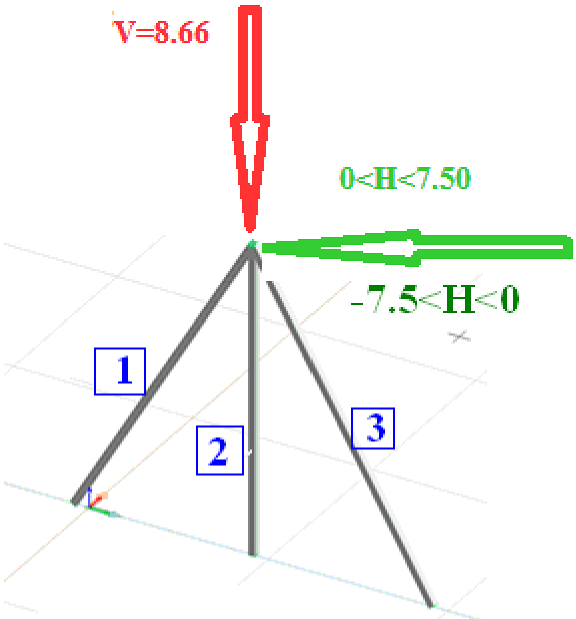

Figure 3: Three-bar truss problem in the case of two point loads at the top

Table 3.a: Comparative values of the optimal 3, 6 and 12-bar structures in the case of two point loads at the top

			3 bar truss		
V	**|H|**	**A1**	**A2**	**A3**	**Vol 3-bar truss**
8,660254	0	0	2,598	0	4,499867998
8,660254	0,5	0,258	2,585	0,258	5,509351338
8,660254	1	0,581	2,434	0,581	6,539811666
8,660254	1,5	0,967	2,152	0,967	7,595373338
8,660254	2	1,407	1,753	1,407	8,664285066
8,660254	2,5	1,90	1,237	1,9	9,742546849
8,660254	3	2,451	0,591	2,451	10,82764203
8,660254	3,5	2,964	0,038	2,964	11,92181793
8,660254	4	3,278	0	3,278	13,112
8,660254	4,5	3,619	0	3,619	14,476
8,660254	5	3,999	0	3,999	15,996
8,660254	5,5	4,419	0	4,419	17,676
8,660254	6	4,879	0	4,879	19,516
8,660254	6,5	5,379	0	5,379	21,516
8,660254	7	5,919	0	5,919	23,676
8,660254	7,5	11,981	0	11,981	47,924

Table 3.b: Comparative values of the optimal 3-6 and 12-bar structures in the case of two point loads at the top

			2 bar truss		
V	**|H|**	**A1**	**Volu of 2-bar truss**	**A2**	**A3**
8,660254	0	2	8	0	2
8,660254	0,5	2,019	8,076	0	2,019
8,660254	1	2,079	8,316	0	2,079
8,660254	1,5	2,179	8,716	0	2,179
8,660254	2	2,319	9,276	0	2,319
8,660254	2,5	2,50	9,996	0	2,499
8,660254	3	2,72	10,88	0	2,72
8,660254	3,5	2,98	11,92	0	2,98
8,660254	4	3,28	13,12	0	3,28
8,660254	4,5	3,619	14,476	0	3,619
8,660254	5	3,999	15,996	0	3,999
8,660254	5,5	4,419	17,676	0	4,419
8,660254	6	4,879	19,516	0	4,879
8,660254	6,5	5,379	21,516	0	5,379
8,660254	7	5,919	23,676	0	5,919
8,660254	7,5	11,981	47,924	0	11,981

According to the papers of Rozvany and Maute [12] or Silva et al [11] the optimal layout is a two leg structure with a well-defined inclination angle if the horizontal force is uncertain. Here a very special case is studied where the initial layout is based on the optimal layout coming from the above cited papers. In addition to the two legs structure an additional vertical leg is considered forming a statically undeterminate structural layout. The problem is a minimum volume design of a three legs structure with constrained compliance – the formulation (eq.1.) is the same and smaller than a given bound)-. The member forces are calculated from the equilibrium equations taking into account the compatibility equations as well. The top load is deterministic with given value, while the top horizontal force is probabilistic. It is modeled on the way that this force can be any value in a given interval (Fig. 3) as it is indicated in the above cited papers [11, 12]. The optimality condition to determine the layout is that the horizontal force can not exceed the expected lower and upper bounds (±7,5).

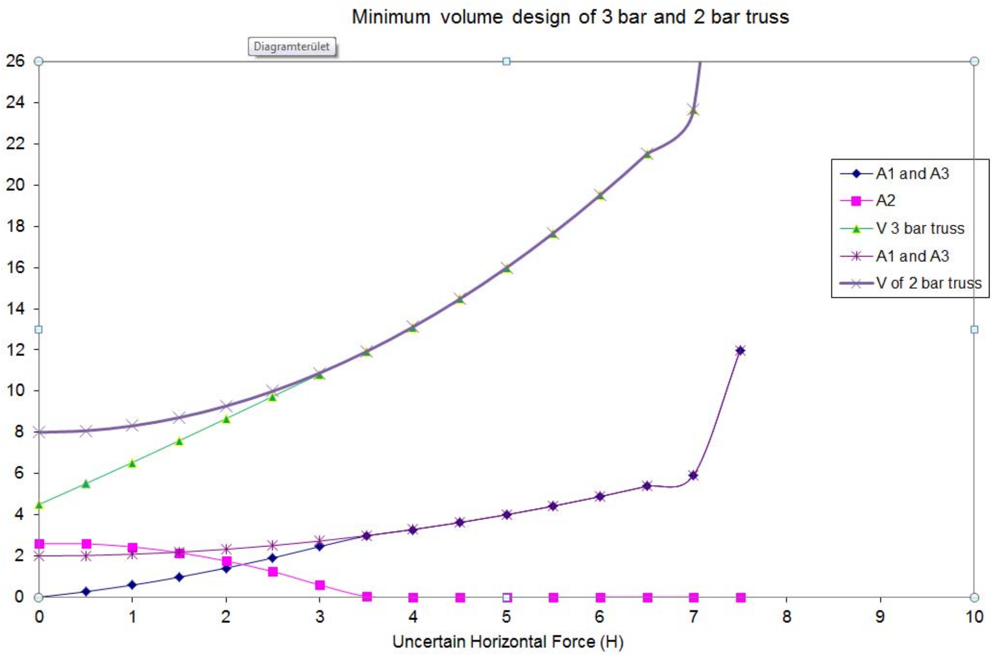

Figure 4: Optimal cross-sectional areas and minimum volumes of three and two bar truss problems

The constrained mathematical programming problem is form with the idea that the unknowns are the cross-sectional areas of the members and the two side legs are in 30 degree inclination angle. There are two load cases (the horizontal force in each case can change its direction). The problem numerically is solved by a sequential quadratic programming algorithm of MATCAD 15.
One can follow the numerical values of the optimal cross-sectional areas in the case of three-bar truss (Table 3.a) and the numerical values of the optimal cross-sectional areas in the case of two-bar structure (Table 3.b), respectively. Graphically these results are presented in Figure 4. One can see that in case of H=0 a vertical bar is the optimal layout while -3.5<H<0<H<3.5 the optimal layout is a three-bar structure. Otherwise the optimal layout is a two legs structure.
A very similar suspicion was presented in the almost forgotten paper of Nagtegaal and Prager [13]. Here the authors investigated the question of the optimal layout in the case of two alternative loads with same point of application. A necessary and sufficient condition for global optimality was derived for the plane truss where the loadings were created on the way that the load factors for plastic collapse under one or the other load were not to exceed a given value. The results were one, two or three bar trusses depending on the loading domains.
The optimal layout problem of a minimum weight truss design problem with a single vertical force load presented by Save [10] in the case of stress constraints. The conclusions of his results and optimal layouts coming from the results obtained from our examples are in good agreement.

5. Probabilistic Compliance Design in the Case of Uncertain Loading Positions
The deterministic compliance design procedure of a linearly elastic 2D structure (disk) in plane stress is known from literature. This topology optimization problem is given for single load as follows:

$$W = \sum_{g=1}^{G} \gamma_g A_g t_g^{\frac{1}{p}} = \min! \tag{2.a}$$

subject to

$$\begin{cases} \mathbf{u}^{\mathbf{T}}\mathbf{F} - C \le 0; \\ -t_g + t_{\min} \le 0; \quad \left(for\ g = 1,...,G \right), \\ t_g - t_{\max} \le 0; \quad \left(for\ g = 1,...,G \right). \end{cases} \tag{2.b-d}$$

Here the ground element thicknesses t_g are the design variables with lower bound $t_{\min}$ and upper bound $t_{\max}$, respectively. By the use of the FE (finite elements) discretization, each ground element (*g= 1,…, G*) contains several sub-elements *(e=1,…,* E_s), whose stiffness coefficients are linear homogeneous functions of the ground element thickness t_g. Furthermore γ_g is the specific weight and A_g the area of the ground element *g*. $\mathbf{u}^{\mathbf{T}}$ is the nodal displacement vector associated with the loading **F**. The displacements u can be calculated from $\mathbf{Ku} = \mathbf{F}$, where **K** is the system stiffness matrix. *p* is the penalty parameter ($p \ge 1$) and the given compliance value is denoted by *C*. The above constrained mathematical programming problem can be solved by the use of an appropriate SIMP algorithm (e.g. Lógó[1]).

5.1. Multiply load cases and uncertain loading magnitudes
The above problem in case of several load cases should be extended by additional compliance constraints representing the independent loadings ($\mathbf{F}^{(i)}$). By the use of a generalized compliance design concept (Lógó [1]) the new constraints

$$\mathrm{P}\left(\mathbf{u}^{(i)T}\mathbf{F}^{(i)} - C \le 0\right) \ge q \tag{3}$$

can be introduced instead of eq.(2.b). Here $0 \langle q \langle 1$ is a given expected probability value what gives information about the possibility of a failure. Following the upperbound theorem of Kataoka [4] eq.(3) can be substituted by the following deterministic expression which is convex and determined for each independent load case:

$$\sum_{j=1}^{n} f_j^{(i)}\, \overline{u}_j^{(i)} - C + \Phi^{-1}\left(q\right)\sqrt{\mathbf{b}^{(i)\mathbf{T}}\mathbf{K}^{(i)}{}_{\mathrm{ov}}\mathbf{b}^{(i)}} \le 0. \tag{4}$$

Here $\overline{u}_j^{(i)} = E(u_j^{(i)}),\ j = 1,...,n$ is the expected value of the displacement under the independent force

$\mathbf{F}^{(i)}$ $(i = 1,...,m)$ in the direction of this load, $\mathbf{b}^{(i)\mathbf{T}} = \left[f_1^{(i)}, f_2^{(i)},..., f_k^{(i)},..., f_n^{(i)} \right]$, $\mathbf{K}_{ov}^{(i)}$ is the covariance matrix of these displacements. The number of the independent load cases depends on the
Then the penalized minimum weight problem subjected to probabilistic compliance constraint due to the uncertain loading magnitude has the form:

$$W = \sum_{g=1}^{G} \gamma_g A_g t_g^{\frac{1}{p}} = \min! \tag{5.a}$$

subject to

$$\begin{cases} \sum_{i=1}^{n} f_i^{(1)} \overline{u}_i^{(1)} - C + \Phi^{-1}(q) \sqrt{\mathbf{b}_i^{(1)\mathbf{T}} \mathbf{K}_{ov}^{(1)} \mathbf{b}_i^{(1)}} \leq 0; \\ \vdots \\ \sum_{i=1}^{n} f_i^{(m)} \overline{u}_i^{(m)} - C + \Phi^{-1}(q) \sqrt{\mathbf{b}_i^{(m)\mathbf{T}} \mathbf{K}_{ov}^{(m)} \mathbf{b}_i^{(m)}} \leq 0; \\ -t_g + t_{\min} \leq 0; \quad (for\ g = 1,...,G), \\ t_g - t_{\max} \leq 0; \quad (for\ g = 1,...,G). \end{cases} \tag{5.b}$$

This type of constrained mathematical programming problem can be solved by using an appropriate optimality criteria algorithm (see e.g. Lógó[5]).

5.2. Uncertain loading positions
Here a simplified mechanical model is created on basis of the original loading domain. Let us consider the design problem given in Figure 5. Since the loading positions are not known precisely an equivalent loading system should be also created around the expected location $\overline{x}_i$ of each force $\mathbf{f}_i$ to perform the simulation. According to the original distribution assumption, the mean value and the standard deviation of the point application are determined by the force system f_{ij} $(j = 1,..,k)$ with the original magnitude f_i - for sake of simplicity and to describe the loading domains- seven points – as "base" points are used with symmetrical adjustment $(f_{i1}, f_{i2}, f_{i3}, f_{i4})$. (The minimum number of the points is three.) Each load is independent and a well-defined probability value w_{ij} $(j = 1,..,k = 7)$ is assigned to them (in practice it can take as design information). The determination of this probability value w_{ij} $(j = 1,..,k)$ is based on the original distribution and it can be calculated with a simple computation. In this way the loading is given by these doubled parameters - w_{ij} $(j = 1,..,k = 7)$, $(f_{i1}, f_{i2}, f_{i3}, f_{i4})$ - and applied as independent load cases. The modified topology design problem is given in Figure 5 if the original load and the supports are located on the same line.

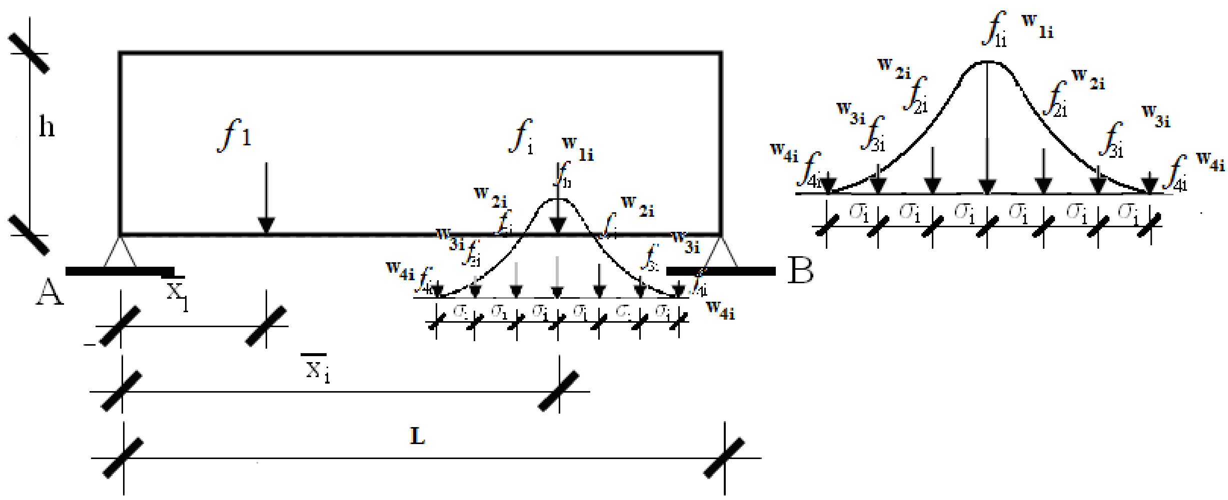

Figure 5: The design domain with the modified loadings and the corresponding probabilities

If the application points and the supports can not be connected with a single line the surrogate model of the loading is based on a force and uncertain moment system at the expected location of the original load.Applying these forces at these “base” points as loads the stochastic design problem becomes a deterministic one after this transformation. By the use of the element f_{ij} $(j = 1,..,k)$ of these force system one by one, the displacement vectors $\mathbf{u}_{ij}$ $(j = 1,..,k)$ can be calculated from the $\mathbf{K}\mathbf{u}_{ij} = \mathbf{f}_{ij}$ linear equations. Since the material is linearly elastic the additive properties of the displacements and the reciprocity theorem can be applied. Using these vectors and the assigned probability values w_{ij} $(j = 1,..,k)$ the expected displacement $\bar{u}_i$ and its variation $D^2(\bar{u}_i)$ can be calculated in the following form:

$$\bar{\mathbf{u}}_i = \sum_{j=1}^{k} \mathbf{u}_{ij} w_{ij} \,; \tag{6.a}$$

$$D_i^2(\bar{u}_i) = \sum_{j=1}^{k} \left(u_{ij}\right)^2 w_{ij} - \bar{u}_i^2 \,. \tag{6.b}$$

These computed values are used to compose the element of the mathematical programming problem eq.(5). Due to the nature of this type of loading the covariance matrix is diagonal.

$$\mathbf{K}_{ov} = \left\langle D_1^2(\bar{u}_1), D_2^2(\bar{u}_2), \ldots, D_n^2(\bar{u}_n) \right\rangle \tag{7}$$

Interchanging the expected compliance calculation by the generalized expected strain energy formulation the penalized minimum weight problem subjected to probabilistic compliance constraint has the form:

$$W = \sum_{g=1}^{G} \gamma_g A_g t_g^{\frac{1}{p}} = \min! \tag{8.a}$$

subject to

$$\begin{cases} \sum_{i=1}^{n} \bar{\mathbf{u}}_i^{(1)T} \mathbf{K} \mathbf{u}_i^{(1)} - C + \Phi^{-1}(q) \sqrt{\mathbf{b}_i^{(1)\mathbf{T}} \mathbf{K}_{ov}^{(1)} \mathbf{b}_i^{(1)}} \le 0; \\ \vdots \\ \sum_{i=1}^{n} \bar{\mathbf{u}}_i^{(m)T} \mathbf{K} \mathbf{u}_i^{(m)} - C + \Phi^{-1}(q) \sqrt{\mathbf{b}_i^{(m)\mathbf{T}} \mathbf{K}_{ov}^{(m)} \mathbf{b}_i^{(m)}} \le 0; \\ -t_g + t_{\min} \le 0; \quad \left(for\ g = 1, \ldots, G \right), \\ t_g - t_{\max} \le 0; \quad \left(for\ g = 1, \ldots, G \right). \end{cases} \tag{8.b-d}$$

6. Numerical example

To demonstrate the method introduced above the example problem of Rozvany and Maute [12] is used to create the base problem (Figure 6.a). The point of application of the vertical load is uncertain. The geometry is given by

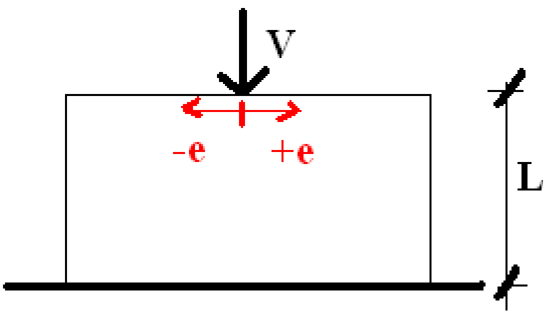

Figure 6.a: Base problem for the SIMP-type solution

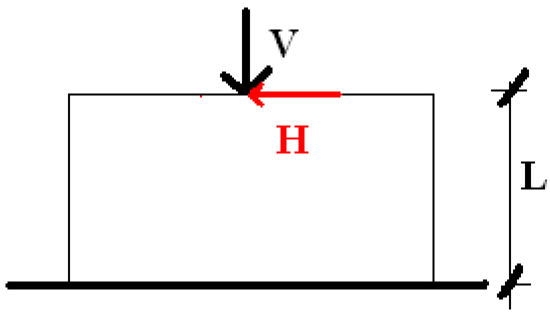

Figure 6.b: Surrogate loading

L=40 while the deterministic magnitude of load is V=50. The values (-e to +e) demonstrate the deviation of the point of application of the force V. The surrogate loading is represented by an additional horizontal force system H corresponding to the "eccentricity e". (Here for demonstrative reason H is 50.) Due to the nature of this problem three independent load cases have to considered. These load cases are: (V=50, H=-50), (V=50, H=0) and (V=50, H=50) –Figure 6.b-. The compliance limit is 60000. The expected probability is q=0,9.
The obtained optimal topology can be seen in Figure 7. One can see that a statically undeterminate structure is the optimal layout. The white lines demonstrate the center line of the truss members. The inclination angle is 36°.

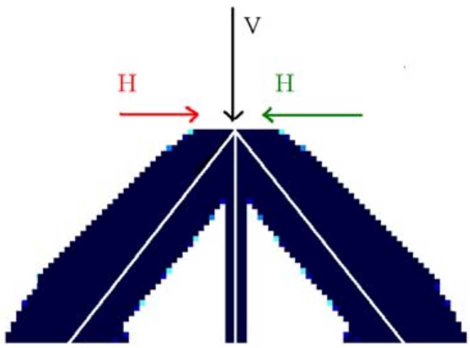

Figure 7: Optimal layout

7. Conclusions

If the load is probabilistic, surrogate deterministic load cases are suggested to model the uncertain point of applications. Minimum three independent load cases need to model the uncertainty connected to an uncertain point of application of the original load. The surrogate loading system is problem dependent.
In case of probabilistic loading the optimal layout can be statically undeterminate structure.
To make more appropriate models need some additional investigations on the topic.

8. Acknowledgements

The present study was supported by the Hungarian National Scientific and Research Foundation (OTKA) (grant K 81185).

9. References

[1] J. Lógó, New Type of Optimality Criteria Method in Case of Probabilistic Loading Conditions, *Mechanics Based Design of Structures and Machines*, 35, 2, 147-162, 2007.
[2] J. Lógó, J., Ghaemi, M. and Vásárhelyi, A. Stochastic compliance constrained topology optimization based on optimality criteria method, *Periodica Polytechnica-Civil Engineering*, 51, 2, 5-10, 2007.
[3] J. Lógó, M. Ghaemi and M. Movahedi Rad, Optimal topologies in case of probabilistic loading: The influence of load correlation, *Mechanics Based Design of Structures and Machines*, 37, 3, 327-348, 2009.
[4] S. Kataoka, A Stochastic Programming Model, *Econometria*, 31, 181-196, 1963.
[5] J. Lógó, SIMP type topology optimization procedure considering uncertain load position, *Periodica Polytechnica-Civil Engineering*, 56, 2, 213-220, 2012.
[6] P.D. Dunning, H.A. Kim and G. Mullineux, Introducing Loading Uncertainty in Topology Optimization, *AIAA Journal*, 49(4), 760-768, 2011.
[7] J.K. Guest and T. Igusa, Structural optimization under uncertain loads and nodal locations, *Comp. Meth. Appl. Mech. Eng.,* 198, 116-124, 2008.
[8] M. Jalalpour, T. Igusa and J.K. Guest, Optimal design of trusses with geometric imperfections: Accounting for global instability, *International Journal of Solids and Structures*, 48(21), 3011-3019, 2011.
[9] J. Barta, On the Minimum Weight of Certain Redundant Structures, *Acta Tech. Acad. Sci. Hung*.; 18, 67-76, 1957.
[9] G. Sved, The Minimum Weight of Certain Redundant Structures, *Australian Journal of Applied Sciences,* 5, 1-9, 1954.
[10] M.A. Save, Remarks on Minimum-Volume Designs of a Three-bar Truss, *Journal of Structural Mechanics,* 11,1,, 101-110, 1983.

[11] M. Silva, D, Tortorelli, J.A. Norato, C. Ha and H.R. Bae, Component and system reliability-based topology optimization using a single loop method, *International Journal for Structural and Multidisciplinary Optimization*, 41, 1, 87-106, 2010
[12] G.I.N. Rozvany and K. Maute. Analytical and numerical solutions for a reliability-based benchmark example.*Structural and Multidisciplinary Optimization*, 43, 6, 745-753, 2011.
[13] J.C. Nagtegaal and W. Prager Optimal layout of a truss for alternative loads, *International Journal of Mechanical Sciences*,15, 7, 583-592, 1973.
[14] G.I.N. Rozvany, On symmetry and non-uniqueness in exact topology optimization, *Structural and Multidisciplinary Optimization*, 43, 3, 297-317, 2011.