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Approximate Fuzzy Structural Analysis Applying Taylor Series and Intervening Variables

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1. Abstract
Fuzzy structural analysis is a well developed tool for uncertainty quantification in computational mechanics. However, its application may be numerically demanding, as it involves the solution of optimization problems in order to determine extrema of the structural response. This contribution explores the application of Taylor series and intervening variables for performing fuzzy structural analysis efficiently. Results presented indicate the application of an intervening variable of the reciprocal type may offer significant improvement in terms of accuracy with respect to traditional approximations.
2. Keywords: fuzzy structural analysis, Taylor series, intervening variable, reciprocal variable.

3. Introduction
The importance of explicitly considering the effects of uncertainties in structural analysis has been widely acknowledged by the engineering community [8, 13]. A number of approaches for uncertainty quantification have been developed within the framework of *classical* probabilities as well as Bayesian techniques, see e.g. [5, 39], etc. Recently, the so-called non traditional approaches for uncertainty quantification have gained considerable attention as well, e.g. [3, 7, 14, 25]. In particular, approaches based on interval analysis and fuzzy analysis have been the subject of active research [29, 30]. It should be noted that irrespective of the approach used to quantify the effects of uncertainty, the application of such procedures is usually much more involved from a numerical viewpoint than performing deterministic analyses. This is due to the fact the structural performance is not quantified by means of a unique, deterministic quantity but by a set of possible outcomes.
In interval analysis (see e.g. [19, 37]), uncertainty in the value of one or more parameters of a model is quantified in terms of bounds. Then, the objective of interval analysis is determining the bounds of a structural response of interest given bounds associated with the unknown input parameters. Fuzzy structural analysis can be interpreted as a sequence of interval analyses as pointed out in [29]. That is, uncertain input parameters are assigned a *membership* which varies between 0 and 1. For each different value of the membership function, the input variables of a model can be characterized by means of intervals. Hence, the structural response can be characterized as an interval as well. In this manner, it is possible to determine the membership for the response function. The latter procedure has been termed in the literature as α-level optimization (see e.g. [29, 31]).
A major challenge for the practical implementation of fuzzy structural analysis using the α-level optimization is the associated numerical costs. For each α-level analyzed (i.e. value of the membership function), it is necessary to determine the minimum and maximum of the structural response given that the uncertain structural parameters lie on a certain interval. Clearly, this is an optimization problem that can be extremely challenging due to issues such as non linearities of the functions involved and the inherent difficulties associated with the determination of global optima [1, 21]. Hence, a number of approaches have been devised in order to overcome this issue, see e.g. [6, 11, 15, 24]. Among these approaches, Taylor series expansion has received considerable attention [10, 17, 26, 37, 27]. This is due to the fact numerical efforts associated with the construction of a Taylor series expansion are limited to a single structural analysis (calculation of stiffness matrix inverse) plus additional assembly and multiplication of structural matrices for calculating sensitivities [20]. Nonetheless, Taylor series may not be always appropriate as they may fail in capturing nonlinearities of the functions being approximated. In view of this issue, this contribution explores the application of Taylor series considering *intervening variables*. The latter variables have been applied customarily in the field of structural optimization (see e.g. [21, 35, 38]) and have also been considered within the framework of structural reliability and classical probabilities [16, 40, 42].

The structure of this paper is the following. First, the basic formulation of the deterministic structural analysis problem and some basic concepts of fuzzy structural analysis are reviewed briefly in Section 4. Section 5 discusses approximate representation of structural responses by means of first- and second-order Taylor expansions. The application of intervening variables (in particular, reciprocal variables) is discussed in section 6. Section 7 presents test examples illustrating the application of Taylor series. Finally Section 8 closes the paper with some conclusions and outlook.

4. Formulation

Assume a linear elastic structure modeled using the FE method [22]. The model comprises N_e elements and N_d degrees-of-freedom (DOFs). In addition, assume there are N_u uncertain parameters $x_i,\ i=1,\ldots,N_u$(such as geometry, loadings, etc.) that are characterized as fuzzy variables. The fuzzy set $\tilde{x}_i,\ i=1,\ldots,N_u$ associated with each of these variables is:

$$\tilde{x}_i = \{(x_i, \mu_{\tilde{x}_i}(x_i)) : (x_i \in X_i) \wedge (\mu_{\tilde{x}_i}(x_i) \in [0,1])\},\ i=1,\ldots,N_u \tag{1}$$

where $\mu_{\tilde{x}_i}(x_i),\ i=1,\ldots,N_u$ represents the membership function associated with the i-th variable and X_i is the set that contains x_i. The membership function ranges between 0 and 1 and is assumed as a convex function in the following. In case $\mu_{\tilde{x}_i}(x_i^*)=0$, then x_i^* does not belong to the fuzzy set; in case $\mu_{\tilde{x}_i}(x_i^*)=1$, x_i^* is a member of the fuzzy set; finally, in case $0<\mu_{\tilde{x}_i}(x_i^*)<1$, membership of x_i^* to the fuzzy set is uncertain.

Consider the uncertain parameters are grouped in a N_u dimensional vector $\boldsymbol{x}=\langle x_1,x_2,\ldots,x_{N_u}\rangle^T$. Then, the equilibrium equation associated with the FE model of the structure is the following.

$$\boldsymbol{K}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) \tag{2}$$

In above equation, $\boldsymbol{K}(\boldsymbol{x})$ is the $N_d\times N_d$ matrix of stiffness, $\boldsymbol{u}(\boldsymbol{x})$ is the $N_d\times 1$ vector of displacements and $\boldsymbol{f}(\boldsymbol{x})$ is the $N_d\times 1$ vector of (equivalent) nodal forces. Note both the stiffness matrix and force vector depend on fuzzy variables. In view of this issue, the components of the displacement vector $\boldsymbol{u}$ are fuzzy variables as well. Undoubtedly, it is of interest characterizing these fuzzy displacements by means of their associated membership functions. The characterization of the latter functions in an exact, analytical form is quite challenging [6, 29]. Hence, these membership functions are represented in a discrete way, i.e. possible displacements values are computed at specific α-cuts, where α denotes the membership level under analysis [6, 29]. The practical implementation of this procedures is as follows. The α-cut associated with the i-th uncertain variable is:

$$\underline{x}_{i,\alpha_k} = \{x_i \in \tilde{x}_i : \mu_{\tilde{x}_i}(x_i) \geq \alpha_k\},\ i=1,\ldots,N_u,\ \alpha_k\in[0,1] \tag{3}$$

where $\alpha_k,\ k=1,\ldots,N_c$ denotes the α-cut value being studied and $\underline{x}_{i,\alpha_k}$ denotes the set of possible values x_i may assume for a given value α_k. Then, the α-cut associated with the n-th displacement is:

$$\underline{u}_{n,\alpha_k} = \left\{u_n : \left(x_i \in \underline{x}_{i,\alpha_k},\ i=1,\ldots,N_u\right) \wedge (u_n = u_n(\boldsymbol{x}))\right\} \tag{4}$$

where $u_n(x_1,x_2,\ldots,x_{N_u})$ is the n-th component of the vector $\boldsymbol{u}(\boldsymbol{x})$ that solves Eq.(2) and $\underline{u}_{n,\alpha_k}$ is the set of possible values u_n assumes for the α-cut value α_k. The interpretation of Eq.(4) is the following: it describes the set that contains all possible values the displacement may assume given the unknown parameters lie within the interval $\underline{x}_{i,\alpha_k}$. This is represented schematically in Fig. 1 where it is assumed $N_u=1$ for the sake of simplicity.

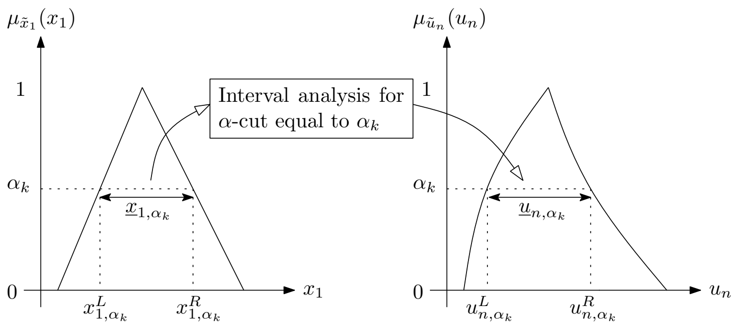

Figure 1: Schematic representation of fuzzy structural analysis considering α-cuts

Note that under the assumption that the sets $\underline{x}_{i,\alpha_k}$ are compact and convex, these sets are fully described by their minimum and maximum value (denoted as x^L_{i,α_k} and x^R_{i,α_k}, respectively, see Fig. 1). Furthermore, under the additional assumption of a continuous mapping function between the unknown input variables $\boldsymbol{x}$ and the output variables $\boldsymbol{u}$ (see Eq.(2)), the set $\underline{u}_{n,\alpha_k}$ is also fully described by its minimum and maximum value (denoted as u^L_{n,α_k} and u^R_{n,α_k}, respectively, see Fig. 1). Hence, the determination of the set $\underline{u}_{n,\alpha_k}$ involves the solution of two optimization problems [6]. That is, u^L_{n,α_k} is the solution of the optimization problem:

$$
\begin{aligned}
u^L_{n,\alpha_k} =& \min_{\boldsymbol{x}} \left(u_n(\boldsymbol{x}) \right) \\
\text{subject to}& \\
& x_i \in \underline{x}_{i,\alpha_k},\ i = 1, \ldots, N_u \\
& \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})
\end{aligned}
\tag{5}
$$

while u^R_{n,α_k} is the solution of:

$$
\begin{aligned}
u^R_{n,\alpha_k} =& \max_{\boldsymbol{x}} \left(u_n(\boldsymbol{x}) \right) \\
\text{subject to}& \\
& x_i \in \underline{x}_{i,\alpha_k},\ i = 1, \ldots, N_u \\
& \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})
\end{aligned}
\tag{6}
$$

The solution of the optimization problems posed in Eqs.(5) and (6) can be quite challenging. From the point of view of optimization, the problems in Eqs.(5) and (6) may possess local optima thus rendering the identification of the global optimum cumbersome. From a numerical viewpoint, the evaluation of the displacement u_n demands performing structural analysis (inverse of stiffness matrix) which can be costly due to the dimension of the model. In order to cope with these issues, different strategies have been proposed in the literature (see e.g. [12, 15, 18, 31, 41]). As already stated above, the focus of this contribution is on the application of strategies based on Taylor series expansion for fuzzy structural analysis [26, 37]. In the following and for the sake of simplicity, uncertainties are assumed to affect the stiffness matrix only (cf. Eq.(2)). This implies the equivalent nodal load vector is modeled as deterministic.

5. Application of Taylor Series

5.1. First-order Taylor Expansion

As discussed previously, the calculation of the displacement vector $\boldsymbol{u}$ is numerically demanding as it involves structural analyses. A possible means for avoiding these demanding analyses consists of approximating the displacement vector by means of a first-order Taylor series. Such an approach is well documented in the literature [27, 36, 37]. For the sake of completeness, the main concepts associated with the implementation of a first-order Taylor expansion are reproduced in the following. For further details, it is referred to the aforementioned references.

The first-order Taylor expansion $\boldsymbol{u}^L$ of the displacement vector is expressed as:

$$
\boldsymbol{u}(\boldsymbol{x}) \approx \boldsymbol{u}^L(\boldsymbol{x}) = \boldsymbol{u}\left(\boldsymbol{x}^0\right) + \sum_{i=1}^{N_u} \boldsymbol{u}_{,i} \left(x_i - x_i^0\right)
\tag{7}
$$

where $\boldsymbol{x}^0$ is the expansion point; x_i^0 is the i-th component of $\boldsymbol{x}^0$ (x_i^0 should belong to the set $\underline{x}_{i,\alpha_k}$); $\boldsymbol{u}\left(\boldsymbol{x}^0\right)$ is the displacement evaluated at the expansion point; and $\boldsymbol{u}_{,i}$ denotes partial derivative of $\boldsymbol{u}$ with respect to x_i evaluated at $\boldsymbol{x}^0$. The latter two vectors are equal to [33]:

$$
\boldsymbol{u}\left(\boldsymbol{x}^0\right) = \boldsymbol{K}\left(\boldsymbol{x}^0\right)^{-1} \boldsymbol{f}
\tag{8}
$$

$$
\boldsymbol{u}_{,i} = \left. \frac{\partial \boldsymbol{u}}{\partial x_i} \right|_{\boldsymbol{x}=\boldsymbol{x}^0} = -\boldsymbol{K}\left(\boldsymbol{x}^0\right)^{-1} \boldsymbol{K}_{,i} \boldsymbol{u}\left(\boldsymbol{x}^0\right),\ i = 1, \ldots, N_u
\tag{9}
$$

where $\boldsymbol{K}_{,i}$ is the partial derivative of the stiffness matrix with respect to x_i evaluated at $\boldsymbol{x}^0$. Considering the approximate representation of the displacement vector in Eq.(7), it is possible to determine an analytical solution for the optimization problems in Eqs.(5) and (6). These solutions are [27]:

$$u_{n,\alpha_k}^L = u_n\left(\boldsymbol{x}^0\right) + \sum_{i=1}^{N_u} u_{n,i} a_{i,\alpha_k},\ n = 1,\ldots,N_u \tag{10}$$

$$u_{n,\alpha_k}^R = u_n\left(\boldsymbol{x}^0\right) + \sum_{i=1}^{N_u} u_{n,i} b_{i,\alpha_k},\ n = 1,\ldots,N_u \tag{11}$$

where the terms a_i and b_i are defined as:

$$a_{i,\alpha_k} = \begin{cases} x_{i,\alpha_k}^R - x_i^0 & \text{if } u_{n,i} \leq 0 \\ x_{i,\alpha_k}^L - x_i^0 & \text{if } u_{n,i} > 0 \end{cases} \tag{12}$$

$$b_{i,\alpha_k} = \begin{cases} x_{i,\alpha_k}^L - x_i^0 & \text{if } u_{n,i} \leq 0 \\ x_{i,\alpha_k}^R - x_i^0 & \text{if } u_{n,i} > 0 \end{cases} \tag{13}$$

From the above description, it is evident that the application of first-order Taylor expansion is numerically efficient as it requires a single matrix factorization. However, its main disadvantage is its accuracy: even if the displacement is mildly nonlinear with respect to the uncertain variables, the bounds calculated using a linear approximation (see Eqs.(10) and (11)) can be far from the actual bounds of the displacement for a particular α-cut.

5.2. Second-order Taylor Expansion

An evident means to improve the accuracy of the approximation in Eq.(7) is including higher order terms, e.g. quadratic, cubic, etc. However, the practical implementation of such strategy may become numerically involved because of two issues. First, the computation of higher order derivatives can be a numerically demanding in case the number of uncertain variables is considerable. Second, the identification of the global optimum for a function involving high order terms can be far from trivial. In view of these issues, the application of Taylor series considering higher order terms has been restricted in the literature to incomplete second-order expansions [10, 17], i.e. expansions that include linear terms and quadratic terms of the type x_i^2 (no cross terms of the type $x_i x_j,\ i \neq j$ are included). In this contribution, a complete second-order Taylor expansion $\boldsymbol{u}^Q(\boldsymbol{x})$ of the displacement vector is considered, i.e.:

$$\boldsymbol{u}(\boldsymbol{x}) \approx \boldsymbol{u}^Q(\boldsymbol{x}) = \boldsymbol{u}\left(\boldsymbol{x}^0\right) + \sum_{i=1}^{N_u} \boldsymbol{u}_{,i}\left(x_i - x_i^0\right) + \frac{1}{2}\sum_{i=1}^{N_u}\sum_{j=1}^{N_u} \boldsymbol{u}_{,ij}\left(x_i - x_i^0\right)\left(x_j - x_j^0\right) \tag{14}$$

where $\boldsymbol{u}_{,ij}$ denotes partial derivative of $\boldsymbol{u}$ with respect to x_i and x_j evaluated at $\boldsymbol{x}^0$. The latter vector is equal to [34]:

$$\boldsymbol{u}_{,ij} = \left.\frac{\partial^2 \boldsymbol{u}}{\partial x_i \partial x_j}\right|_{\boldsymbol{x}=\boldsymbol{x}^0} = -\boldsymbol{K}\left(\boldsymbol{x}^0\right)^{-1}\left(\boldsymbol{K}_{,i}\boldsymbol{u}_{,j} + \boldsymbol{K}_{,j}\boldsymbol{u}_{,i} + \boldsymbol{K}_{,ij}\boldsymbol{u}\left(\boldsymbol{x}^0\right)\right),\ i,j = 1,\ldots,N_u \tag{15}$$

where $\boldsymbol{K}_{,ij}$ is the partial derivative of the stiffness matrix with respect to x_i and x_j evaluated at $\boldsymbol{x}^0$. In order to determine the values of u_{n,α_k}^L and u_{n,α_k}^R based on the approximation $u_n^Q(\boldsymbol{x})$, it is necessary to solve the following optimization problems.

$$u_{n,\alpha_k}^L = \min_{\boldsymbol{x}}\left(u_n^Q(\boldsymbol{x})\right) \tag{16}$$

subject to

$$x_i \in \underline{x}_{i,\alpha_k},\ i = 1,\ldots,N_u$$

$$u_{n,\alpha_k}^R = \max_{\boldsymbol{x}}\left(u_n^Q(\boldsymbol{x})\right) = \min_{\boldsymbol{x}}\left(-u_n^Q(\boldsymbol{x})\right) \tag{17}$$

subject to

$$x_i \in \underline{x}_{i,\alpha_k},\ i = 1,\ldots,N_u$$

In Eq.(17), recall maximizing a function is equal to minimizing its negative value [1, 21].

The optimization problems in Eqs.(16) and (17) can be solved using methods of quadratic programming

[32]. Problems of quadratic programming are challenging to solve whenever the Hessian matrix associated with the function being minimized is not positive semi-definite. It should be noted the problems in Eqs.(16) and (17) fall into the latter category: even if the Hessian matrix associated with any of the optimization problems in Eqs. (16) and (17) is positive semi-definite (thus ensuring that optimization problem is convex), the remaining problem would have an associated Hessian matrix which is negative semi-definite. This is due to the fact the optimization problems in Eqs.(16) and (17) are identical except for the fact one seeks the minimum of $u_n^Q(\boldsymbol{x})$ while the other seeks the minimum of $-u_n^Q(\boldsymbol{x})$. The only case that would escape this classification is the case where the Hessian matrix associated with $u_n^Q(\boldsymbol{x})$ is equal to zero. For the latter case, the problem actually reduces to problem involving linear terms only and hence the approach described in Section 5.1 is applicable.
From the discussion above, it is seen that at least one of the optimization problems in Eqs.(16) and (17) is non-convex. Hence, for solving these optimization problems and determining their global optima, it is necessary to resort to special techniques. In particular, non-convex quadratic programming problems are known to be NP-hard [43, 44]. The solution strategy applied in this contribution consists in combining the so-called branch-and-bound technique with relaxations of the original optimization problem [2, 9].
It is important to note that the solution of the optimization problems of Eqs.(16) and (17) can become numerically involved in view of the arguments discussed previously. It may even be the case that the solution of optimization problems associated with the quadratic approximation of the displacement may be more numerically demanding than solving the *exact* optimization problem (where no approximations of the displacement vector are introduced). Nonetheless, the application of second-order Taylor expansions of the displacement vector is still investigated in this contribution in order to assess its accuracy when compared to the linear case and the case where intervening variables are considered. However, it is explicitly acknowledged that the application of second-order Taylor expansions could potentially imply a considerable numerical burden that may not be worthwhile.

6. Application of First-Order Taylor Series and Intervening Variable of Reciprocal Type

A possible means for improving the the accuracy associated with a first-order Taylor expansion is the application of *intervening* variables. Such variables are applied frequently within the field of structural optimization (see e.g. [21, 28, 35, 38]). However, its application within uncertainty quantification has remained relatively unexplored except for few efforts documented in the literature [16, 40, 42].
Assume intervening variables are defined such that $y_i = y_i(x_i),\ i = 1,\ldots,N_u$. Then, the first-order Taylor expansion of the displacement with respect to these intervening variables (denoted as $\boldsymbol{u}^I(\boldsymbol{x})$) is:

$$\boldsymbol{u}(\boldsymbol{x}) \approx \boldsymbol{u}^I\left(\boldsymbol{y}\left(\boldsymbol{x}\right)\right) = \boldsymbol{u}\left(\boldsymbol{y}\left(\boldsymbol{x}^0\right)\right) + \sum_{i=1}^{N_u} \left.\frac{\partial \boldsymbol{u}}{\partial y_i}\right|_{\boldsymbol{y}=\boldsymbol{y}(\boldsymbol{x}^0)} \left(y_i(x_i) - y_i\left(x_i^0\right)\right) \tag{18}$$

Different types of intervening variables have been proposed in the literature [38, 40]. One of the most commonly used intervening variables is the *reciprocal* as it may lead to an exact representation of the structural response for certain classes of problems [23, 38]. Hence, under the assumption that the set $\underline{x}_{i,\alpha_k}$ excludes the value 0, the intervening variable $y_i(x_i)$ is defined as:

$$y_i(x_i) = \frac{1}{x_i},\ i = 1,\ldots,N_u \tag{19}$$

Replacing Eq.(19) in (18) yields the following expression for the first-order Taylor expansion considering intervening variables of the reciprocal type (denoted in the following as $\boldsymbol{u}^R(\boldsymbol{x})$).

$$\boldsymbol{u}(\boldsymbol{x}) \approx \boldsymbol{u}^R(\boldsymbol{x}) = \boldsymbol{u}\left(\boldsymbol{x}^0\right) + \sum_{i=1}^{N_u} \boldsymbol{u}_{,i} x_i^0 \left(1 - \frac{x_i^0}{x_i}\right) \tag{20}$$

Based on the above approximation $\boldsymbol{u}^R(\boldsymbol{x})$, the analytical solution for the optimization problems in Eqs.(5) and (6) is the following.

$$u_{n,\alpha_k}^L = u_n\left(\boldsymbol{x}^0\right) + \sum_{i=1}^{N_u} u_{n,i} c_i,\ n = 1,\ldots,N_u \tag{21}$$

$$u_{n,\alpha_k}^R = u_n\left(\boldsymbol{x}^0\right) + \sum_{i=1}^{N_u} u_{n,i} d_i,\ n = 1,\ldots,N_u \tag{22}$$

where the terms c_i and d_i are defined as:

$$c_{i,\alpha_k} = \begin{cases} x_i^0 \left(1 - \frac{x_i^0}{x_{i,\alpha_k}^R}\right) & \text{if } u_{n,i} \leq 0 \\ x_i^0 \left(1 - \frac{x_i^0}{x_{i,\alpha_k}^L}\right) & \text{if } u_{n,i} > 0 \end{cases} \tag{23}$$

$$d_{i,\alpha_k} = \begin{cases} x_i^0 \left(1 - \frac{x_i^0}{x_{i,\alpha_k}^L}\right) & \text{if } u_{n,i} \leq 0 \\ x_i^0 \left(1 - \frac{x_i^0}{x_{i,\alpha_k}^R}\right) & \text{if } u_{n,i} > 0 \end{cases} \tag{24}$$

As noted from the above equations, the application of the approximation $\boldsymbol{u}^R(\boldsymbol{x})$ involves numerical efforts which are equivalent to those associated to $\boldsymbol{u}^L(\boldsymbol{x})$. That is, both approaches require a single factorization of the stiffness matrix. However, the major advantage of the approximation $\boldsymbol{u}^R(\boldsymbol{x})$ over $\boldsymbol{u}^L(\boldsymbol{x})$ is that it may reproduce - to some extent - the eventual nonlinear behavior of $\boldsymbol{u}(\boldsymbol{x})$. In fact, for certain types of problems, $\boldsymbol{u}^R(\boldsymbol{x})$ may even approximate $\boldsymbol{u}(\boldsymbol{x})$ exactly as already stated above.

7. Examples

7.1. Example 1: Statically Determinate Truss Structure

The first example consists of the 13-bar statically determinate truss depicted in Fig. 2. The truss is subjected to 3 external loadings. All bars of the truss possess a Young's modulus equal to $E = 2 \times 10^{11}$ [Pa]. In addition, for each bar, the cross section area is modeled as a fuzzy variable. That is, the model comprises a total of 13 fuzzy variables ($N_u = 13$). The membership function associated with each of these fuzzy variables is shown in Fig. 3. The objective is determining the membership function associated with the vertical displacement of node A.

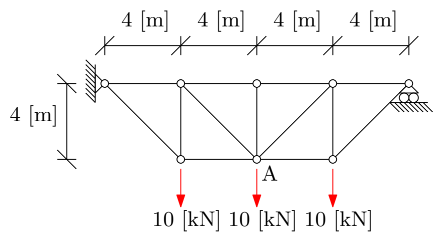

Figure 2: Schematic representation of structure considered in example 1

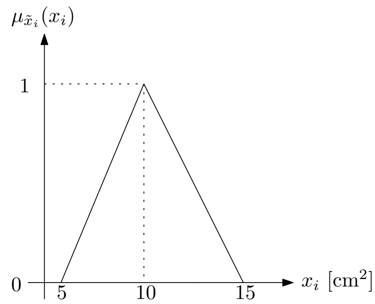

Figure 3: Membership function associated with cross section area of each bar

In order to estimate the sought membership function, four different approaches are employed. These approaches involve first- and second-order Taylor expansions considering no intervening variables, first-order Taylor expansion considering reciprocal intervening variables and direct solution of Eqs.(5) and (6) by means of an optimization algorithm. These approaches are labeled as 'T1', 'T2', 'R' and 'D' in the following. It should be noted that for approaches 'T1', 'T2' and 'R', the expansion point is selected as $\boldsymbol{x}^0 = \langle 10, 10, \ldots, 10 \rangle^T$ [cm^2]. For the case of approach 'D', a sequential quadratic programming algorithm is used for solving the optimization problem [32]. In order to avoid determining local minima, the algorithm is executed several times from random starting points.

The estimated membership functions using each of the 4 approaches described above are summarized in Fig. (4). As noted from the figure, there is a perfect match between the membership function generated using the reciprocal intervening variable ('R') and direct optimization ('D'). This was an expected result: as the truss under study is statically determinate, the application of a reciprocal intervening variable leads to an approximate representation that is actually equal to the exact displacement. In addition, the approximations considering Taylor expansions produce results which are approximate. Naturally, the accuracy of the second-order expansion ('T2') is higher than the accuracy of the first-order expansion ('T1'). However, numerical costs associated to 'T2' are much higher than those associated with 'T1'. Note numerical efforts associated with direct optimization ('D') are also considerable while the costs associated with first-order expansions with and without intervening variables are equal and substantially smaller than the previous cases.

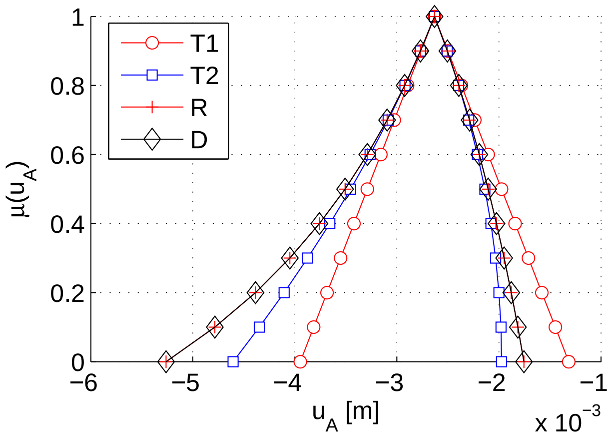

Figure 4: Membership function associated with vertical displacement of node A

7.2. Example 1: Statically Indeterminate Truss Structure
The second example consists of the 21-bar statically indeterminate truss depicted in Fig. 5. The reason for choosing an indeterminate structure is analyzing the performance of a first-order approximation considering intervening reciprocal variables in order to assess (qualitatively) how its accuracy decreases with respect to the case where a determinate structure is analyzed.
The parameters of the 21-bar truss replicate those of example 1 of this contribution except that in this case, the problem involves a total of $N_u = 21$ fuzzy variables related with the cross section area of the bars. The membership function associated with each of these areas is represented schematically in Fig. 3. The objective is determining the membership function associated with the horizontal displacement of node B. The sought membership function is determined using the same four approaches used in example 1. The results obtained are presented in Fig. 6.

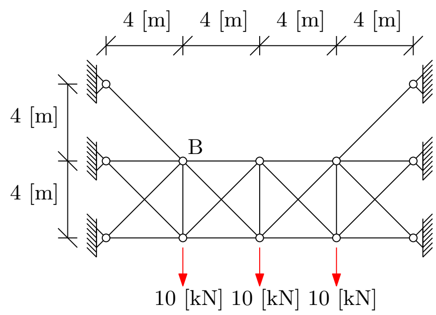

Figure 5: Schematic representation of structure considered in example 2

The results obtained are most interesting. For almost all α-cuts analyzed, the approximation that is closest to the results provided by direct optimization is the first-order Taylor expansion considering reciprocal variables. The second-order Taylor expansion provides results which are quite close to those provided by the approximation considering reciprocal variables. Finally, the first-order expansion considering no intervening variables provides the worst approximation when compared with direct optimization.
It is interesting to note that although the structure analyzed is not statically determinate, the approximation involving reciprocal intervening variables still outperforms the other approximations considered. Such a behavior has already been recognized within the context of structural optimization [4].

8. Conclusions

The results presented in this contribution suggest the application of intervening variables of the reciprocal type may bring substantial advantages for performing fuzzy structural analysis, i.e. propagating the uncertainty from input variables to output responses (in this case, displacements) where uncertainty

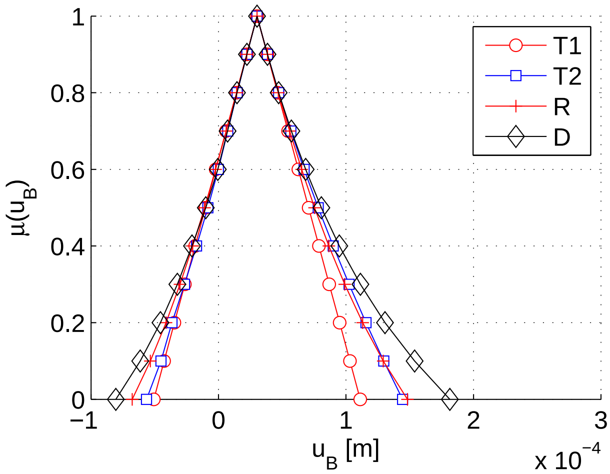

Figure 6: Membership function associated with horizontal displacement of node B

is quantified in terms of membership functions. The examples analyzed indicate a first-order Taylor expansion of the displacement considering reciprocal intervening variables outperforms both first- and second-order Taylor expansions considering no intervening variables. In addition, numerical costs associated with the use of a first-order Taylor expansion do not change irrespective of the use of a reciprocal intervening variable or no intervening variable at all. On the contrary, numerical efforts do increase considerably for the case where a second-order Taylor expansion is employed.
The main reason behind the successful application of a reciprocal intervening variable is its capacity of capturing the nonlinear behavior of the exact structural response. However, it must be acknowledged that for problems of practical interest, capturing the nonlinear behavior of the response is only one side of the problem. The other side is coping with the presence of extrema of the structural response which take place for values of the uncertain parameters which lie *within* the bounds of these parameters (and not *on* these bounds). Note the first-order Taylor expansion considering reciprocal intervening variables presented in Eqs.(21), (22), (23) and (24) computes the extrema of the response assuming these extrema take place in the boundary of the region containing the uncertain parameters. However, there is no warranty that this actually takes place (except for specific cases such as the one analyzed in Example 1). Although the results presented are promising it should be kept in mind the examples analyzed are of limited scope. Future research efforts will aim at considering different types of structures. In the same manner, different types of intervening variables (other than reciprocal) could be considered as well. Finally, other types of approximation (besides a first-order Taylor expansion) should be considered for accounting for those cases where the extrema of the response do not lie on the boundary of the region containing the uncertain parameters. In fact, efforts on researching these issues are currently under way.

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