10th World Congress on Structural and Multidisciplinary Optimization
May 19 - 24, 2013, Orlando, Florida, USA

Linear Programming Approach to Design of Link Mechanisms of Partially Rigid Frames

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1. Abstract
A simple systematic approach is presented for designing a spatial link mechanism with partially rigid joints. An infinitesimal mechanism that undergoes the desired deformation is obtained by solving a linear programming problem that maximizes the load factor under the equilibrium condition and upper- and lower-bound constraints on the member-end forces of a given frame structure. Similarity between this limit analysis problem and a linear programming problem to obtain a sparse solution, including a few nonzero variables, is discussed comparing several formulations of linear programming problems. Geometrically nonlinear large-deformation analysis is then carried out to verify geometrically nonlinear behavior of the mechanism obtained by solving a presented linear programming problem. It is shown in the numerical examples that various mechanisms can be easily found using the proposed method.

2. Keywords: Link mechanism; deployable structure; partially rigid joint; plastic limit analysis; linear programming.

3. Introduction
Link mechanism is used mainly in the field of mechanical engineering to modify the amplitude and/or direction of displacement. For designing mechanisms, many methods have been proposed mainly based on analytical formulations that are applicable to mechanisms with small degrees of freedom of displacements [1].

Recently, several computational optimization approaches have been developed to design of link mechanisms [2]. Most of them are based on the ground structure approach, i.e., they first prepare a ground structure which has many bars and joints and next remove unnecessary bars or joints as a result of optimization. However, only a limited number of initial solutions lead to a mechanism that exhibits the desired deformation. Furthermore, a feasible solution, where the output displacement is in the specified direction, is found only if the initial solution is selected appropriately. A graph theoretical approach has been presented to enumerate mechanisms from a given ground structure [3]. An enumeration approach has also been presented for generating statically determinate trusses which can be used as ground structures with small number of bars [4].

Most of the numerical approaches are applied to planar mechanisms. For three-dimensional mechanisms, it is not practically acceptable to assign ideal pin-joints for all nodes to rotate around three axes. Therefore, it is desired to design a three-dimensional mechanism with partially rigid joints. The authors developed a method for generating deployable structures composed of bars connected with partially rigid joints [5]. There, the formulation of plastic limit analysis was used to find an infinitesimal mechanism that exhibits a desired deformation. However, it is difficult to find a mechanism such that the output node moves with the desired magnitude in the specified direction as a result of displacement of input node.

In this study several linear programming (LP) problems are presented for obtaining a link mechanism that has moderately small numbers of hinges and members. Such problems are related with those for finding a *sparse* solution of a system of linear equations, where a sparse solution means a vector having small number of nonzero components. Mechanisms are found by solving one of the LPs. In the numerical experiments, dependence of the solution on problem parameters is discussed. A solution is also compared with the one obtained by solving the conventional plastic limit analysis problem. Finally, a three-dimensional mechanism is generated to ensure applicability of the proposed approach to design of deployable structures.

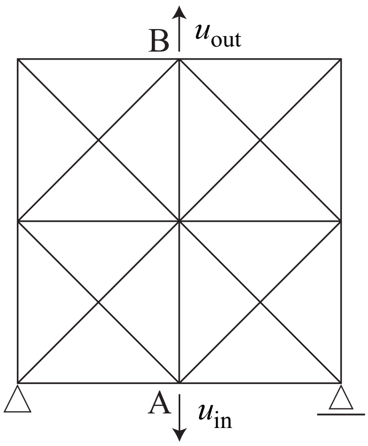


Figure 1: Model 1.

4. Design problem of link mechanisms as partially rigid frames

Like a conventional ground structure method, we prepare a frame structure in the two- or three-dimensional space. The frame structure consists of m members and locations of the nodes are specified. This structure is used as an initial solution for design process of link mechanisms. We adopt the Euler–Bernoulli beam elements for modeling the members. We assume small deformation except in the geometrically nonlinear large-deformation analysis performed in section 7.4.

Let $\mathbf{u} \in \mathbb{R}^d$ denote the displacement vector of the frame structure, where d is the number of degrees of freedom of displacements. We use $\mathbf{c} = (c_1, \dots, c_n)^{\mathrm{T}} \in \mathbb{R}^n$ to denote the generalized strain vector, where $n = 3m$ for a planar frame structure and $n = 6m$ for a spatial frame structure. The compatibility relation between c_i and $\mathbf{u}$ can be written as

$$c_i = \mathbf{h}_i^{\mathrm{T}} \mathbf{u}, \tag{1}$$

where $\mathbf{h}_i \in \mathbb{R}^d$ is a constant vector. Note that matrix $\mathbf{H} \in \mathbb{R}^{d \times n}$ defined by

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_1 \mid \mathbf{h}_2 \mid \cdots \mid \mathbf{h}_n \end{bmatrix} \tag{2}$$

is the equilibrium matrix.

Fig. 1 shows an example of initial frame structure. We attempt to generate a link mechanism by appropriately releasing some of member-end forces and removing some members. Each node with the released member-end forces yields a partially rigid joint. A link mechanism with only partially rigid joints has an advantage in manufacturability compared with the one involving pin-joints, because it is difficult to realize ideal pin-joints that can freely rotate around all the three axes in the three-dimensional space.

In Fig. 1, we suppose that node A, called the *input node*, is subjected to a prescribed displacement, $\bar{u}_{\text{in}}$ (> 0), in the specified direction. We denote by u_{out} the displacement of node B, called the *output node*, in the specified direction. Then we look for a link mechanism that satisfies $u_{\text{out}} \geq \bar{u}_{\text{out}}$, where $\bar{u}_{\text{out}} > 0$ is a specified lower bound. Such a mechanism satisfies

$$c_i = \mathbf{h}_i^{\mathrm{T}} \mathbf{u}, \quad i = 1, \dots, n, \tag{3a}$$

$$u_{\text{in}} = \bar{u}_{\text{in}}, \tag{3b}$$

$$u_{\text{out}} \geq \bar{u}_{\text{out}}. \tag{3c}$$

Here, (3a) is the compatibility relation; see (1). Note that a system of linear equations in (3a) is underdetermined, because $\mathbf{c} = (c_1, \dots, c_n)^{\mathrm{T}}$ and $\mathbf{u}$ are unknown variables. A solution $(\mathbf{c}, \mathbf{u})$ of (3) represents the deformation of a generated link mechanism, where $\mathbf{u}$ is its displacement vector when no external force is applied and the displacement of the input node is prescribed. The generalized strain, c_i, represents the internal deformation corresponding to $\mathbf{u}$. In other words, if $c_i \neq 0$, then the corresponding internal force is released to generate the link mechanism. Conversely, if $c_i = 0$, then the corresponding degree of freedom of displacement is fixed rigidly so as to retain an internal force. More precisely, if $c_i \neq 0$ corresponds to elongation of a member, then the member itself is removed. Alternatively, if $c_i \neq 0$

corresponds to rotation of a joint, then the constraint on the rotation around the corresponding axis is released to make a partially rigid joint.

Thus a candidate of link mechanism can be obtained as a solution of (3). Since (3a) is underdetermined, system (3) has infinitely many, if any, solutions. Suppose that, at one of those solutions, many components of $\mathbf{c}$ are nonzero. This means that many degrees of member-end forces are released. Usually such a link mechanism is not suitable for practical use, because it has a large degree of kinematical indeterminacy. It is desired that a mechanism has a few, possibly only one, mode of inextensional deformations from a practical point of view. Such a link mechanism corresponds to a solution with sparse $\mathbf{c}$. This motivates us to minimize the number of nonzero components of $\mathbf{c}$ under the constraints in (3).

For vector $\mathbf{c} \in \mathbb{R}^n$, define $\mathrm{supp}(\mathbf{c}) \subseteq \{1, \dots, n\}$ by

$$\mathrm{supp}(\mathbf{c}) = \{i \in \{1, 2, \dots, n\} \mid c_i \neq 0\}.$$

We use $|\mathrm{supp}(\mathbf{c})|$ to denote the cardinality of $\mathrm{supp}(\mathbf{c})$, i.e., $|\mathrm{supp}(\mathbf{c})|$ is the number of nonzero components of $\mathbf{c}$. The problem of finding the sparsest solution $\mathbf{c}$ to (3) is formulated as

$$\begin{aligned} \min \quad & |\mathrm{supp}(\mathbf{c})| && (4a) \\ \mathrm{s.t.} \quad & c_i = \mathbf{h}_i^{\mathrm{T}} \mathbf{u}, \quad i = 1, \dots, n, && (4b) \\ & u_{\mathrm{in}} = \bar{u}_{\mathrm{in}}, && (4c) \\ & u_{\mathrm{out}} \geq \bar{u}_{\mathrm{out}}. && (4d) \end{aligned}$$

Unfortunately, it is difficult to solve this optimization problem. In the next section, we introduce a closely related optimization problem and an efficient heuristic method in literature.

5. Sparse solutions of linear equations

Problem (4) is closely related to a problem of finding a sparse solution of a system of linear equations. The latter problem has various applications in, e.g., coding theory and machine learning. Usually it is difficult to deal with an objective function and/or constraints including $|\mathrm{supp}(\cdot)|$. Rather, many heuristic methods for finding sparse solutions have been proposed. It is known that, among them, methods using ℓ_1-norm are efficient [6–10]. In this section we overview two examples in literature.

The first example is a problem of finding a sparse solution of an underdetermined system of linear equations. Exposition below follows Matoušek and Gärtner [11, section 8.5]. Consider a system of linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

where $\mathbf{A} \in \mathbb{R}^{p \times q}$, $\mathbf{b} \in \mathbb{R}^p$, and $p < q$. This system has infinitely many solutions. Among them, suppose that we are interested in a solution $\mathbf{x}$ with at most r nonzero components, where r is a specified integer. This problem is formally written as

$$\begin{aligned} & \mathbf{A}\mathbf{x} = \mathbf{b}, && (5a) \\ & |\mathrm{supp}(\mathbf{x})| \leq r. && (5b) \end{aligned}$$

Problem (5) has applications in, e.g., error correction in a coding problem. In a coding problem, $\mathbf{x}$ corresponds to an error vector and r is the upper bound for the number of corrupted components.

Solving (5) is difficult in general. However, it is known that a sparse solution of (5a) can be obtained by minimizing an appropriate norm of $\mathbf{x}$. Actually, minimization of the ℓ_1-norm of $\mathbf{x}$, denoted $\|\mathbf{x}\|_1 = |x_1| + |x_2| + \dots + |x_q|$, typically yields a solution with only a few nonzero components [7]. This problem, called *basis pursuit*, is written as

$$\begin{aligned} \min \quad & \sum_{j=1}^{q} |x_j| && (6a) \\ \mathrm{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}. && (6b) \end{aligned}$$

This problem can be recast as an LP problem, and hence is solved efficiently.

The second example stems from regression analysis. Let $x_1, \dots, x_q$ and y be explanatory variables and a response variable, respectively. In the linear regression analysis, we attempt to find $\beta_0, \beta_1, \dots, \beta_q \in \mathbb{R}$

such that

$$y \simeq \beta_0 + \sum_{j=1}^{q} \beta_j x_j.$$

For example, the least square estimate is the optimal solution of the following problem:

$$\min \quad \sum_{i=1}^{p} \Big(y_i - \beta_0 - \sum_{j=1}^{q} \beta_j X_{ij} \Big)^2. \tag{7}$$

Here, $\beta_0, \beta_1, \dots, \beta_q$ are variables to be optimized. For several reasons, a sparse solution is often desired in regression analysis. One of well-known methods to find a sparse solution is the so-called LASSO (least absolute shrinkage and selection operator) [9, 10]. In LASSO we add the ℓ_1 penalty function to (7) as

$$\min \quad \sum_{i=1}^{p} \Big(y_i - \beta_0 - \sum_{j=1}^{q} \beta_j X_{ij} \Big)^2 + \omega \sum_{j=1}^{q} |\beta_j|, \tag{8}$$

where $\omega > 0$ is a constant weight. As ω becomes greater, the solution is expected to become sparser.

Thus LASSO solves ℓ_1-regularization of the least square problem. Other regularization methods have been proposed for, e.g., avoiding over-fitting. Among them, making use of the ℓ_2 penalty function is known as the Tikhonov regularization. This idea is sometimes used in optimization of continua to remedy ill-posedness of the topology optimization problem [12].

6. Finding mechanisms as sparse solutions

As explained in section 4, ℓ_1-norm minimization often yields a sparse solution. On the other hand, as discussed in section 3, for designing a practical link mechanism we attempt to find a solution of (3) with sparse $\mathbf{c}$. It may be natural to expect that such a sparse solution can be obtained by minimizing the ℓ_1-norm of $\mathbf{c}$ under (3). As a slight generalization, we consider the weighted ℓ_1-norm of $\mathbf{c}$. This optimization problem, which is much easier than (4) as discussed below, is formulated as

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} |w_i c_i| && \text{(9a)} \\ \text{s.t.} \quad & c_i = \mathbf{h}_i^{\mathrm{T}} \mathbf{u}, \quad i = 1, \dots, n, && \text{(9b)} \\ & u_{\mathrm{in}} = \bar{u}_{\mathrm{in}}, && \text{(9c)} \\ & u_{\mathrm{out}} \ge \bar{u}_{\mathrm{out}}. && \text{(9d)} \end{aligned}$$

Here, $w_i > 0$ $(i = 1, \dots, n)$ are constant weights. For instance, if we set w_i to a large value, then c_i is expected to become 0 at the optimal solution. Therefore, we use a large value for w_i when we do not expect to release a member-end force corresponding to c_i.

We can solve problem (9) as an LP problem. To see this, we introduce additional variables $\gamma_1, \dots, \gamma_n$ that serve as upper bounds for $|c_1|, \dots, |c_n|$. Then problem (9) can be rewritten as

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} w_i \gamma_i && \text{(10a)} \\ \text{s.t.} \quad & -\gamma_i \le \mathbf{h}_i^{\mathrm{T}} \mathbf{u} \le \gamma_i, \quad i = 1, \dots, n, && \text{(10b)} \\ & u_{\mathrm{in}} = \bar{u}_{\mathrm{in}}, && \text{(10c)} \\ & u_{\mathrm{out}} \ge \bar{u}_{\mathrm{out}}. && \text{(10d)} \end{aligned}$$

This is clearly an LP problem in variables $\mathbf{u}$ and $\gamma_1, \dots, \gamma_n$.

We next discuss a similarity that exists between problem (10) and the conventional plastic limit analysis. This similarity can be viewed clearly in the dual problem of (10). For simplicity of presentation, define $\mathbf{p}_{\mathrm{in}} \in \mathbb{R}^d$ as a vector such that the component corresponding to the degree of u_{in} is equal to 1 and all the other components are 0. Similarly, $\mathbf{p}_{\mathrm{out}} \in \mathbb{R}^d$ is the vector such that the component corresponding to the degree of $u_{\rm out}$ is equal to 1 and all the other components are 0, i.e.,

$$\mathbf{p}_{\rm in} = \begin{bmatrix} 1 \\ \hline 0 \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \leftarrow u_{\rm in} \\ \\ \\ \\ \\ \end{matrix}, \qquad \mathbf{p}_{\rm out} = \begin{bmatrix} 0 \\ \hline 1 \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \leftarrow u_{\rm out} \\ \\ \\ \\ \end{matrix}.$$

From the standard duality theory of LP, the dual problem of problem (10) is given by

$$\begin{aligned} \max \quad & \bar{u}_{\rm in}\lambda_{\rm in} + \bar{u}_{\rm out}\lambda_{\rm out} && \text{(11a)} \\ \mathrm{s.\,t.} \quad & \sum_{i=1}^{n} y_i \mathbf{h}_i = \lambda_{\rm in}\mathbf{p}_{\rm in} + \lambda_{\rm out}\mathbf{p}_{\rm out}, && \text{(11b)} \\ & w_i \geq |y_i|, \quad i = 1, \dots, n, && \text{(11c)} \\ & \lambda_{\rm out} \geq 0, && \text{(11d)} \end{aligned}$$

where $\lambda_{\rm in} \in \mathbb{R}$, $\lambda_{\rm out} \in \mathbb{R}$, and $\mathbf{y} \in \mathbb{R}^n$ are variables to be optimized. Note that the primal problem, (9), is always feasible, because for any $\mathbf{u}$ satisfying (9c) and (9d) we can define $\mathbf{c}$ by (9b). Moreover, the dual problem, (11), is also always feasible; for instance, $\lambda_{\rm in} = \lambda_{\rm out} = 0$ and $\mathbf{y} = \mathbf{0}$ are feasible for this problem. Therefore, the strong duality of LP guarantees that problems (10) and (11) have optimal solutions and that their optimal values coincide.

Problem (11) is analogous to the conventional plastic limit analysis problem based on the lower-bound theorem; see, e.g., Jirásek and Bažant [13] for plastic limit analysis. Constraint (11b) is regarded as the force-balance equation, because $\mathbf{H}$ defined by (2) is the equilibrium matrix. Here, y_i corresponds to the generalized stress that is work-conjugate to c_i. The vector on the right-hand side of (11b) is considered the external load, where $\lambda_{\rm in}$ and $\lambda_{\rm out}$ correspond to the loading parameters. Constraint (11c) is analogous to the yield condition, where w_i corresponds to the absolute value of the yield stress. In this way, problem (11) can be regarded as the maximization of the load factor under the constraints of the force-balance equation and the yield conditions. By controlling parameters $w_1, \dots, w_n$ and $\bar{u}_{\rm out}/\bar{u}_{\rm in}$, it may be possible to obtain a useful mechanism that has a small degree of kinematical indeterminacy. Note that the mechanism obtained by this procedure may be either infinitesimal or finite, because issues of geometrical nonlinearity are not taken into account.

As a variant of problem (9), we may transfer constraint (10d) to the objective function to obtain the following problem:

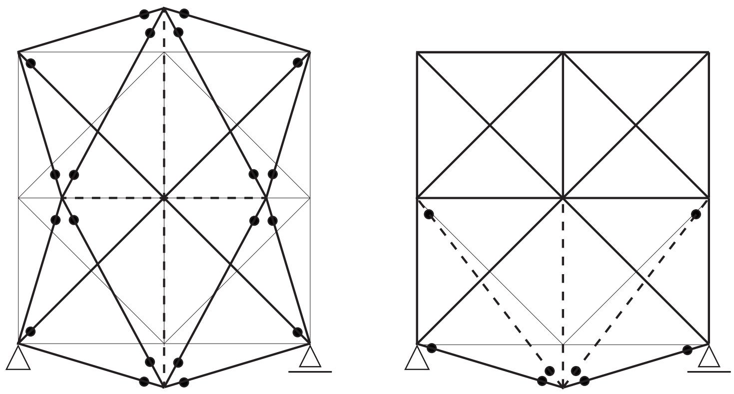
$$\begin{aligned} \min \quad & -u_{\rm out} + \alpha \sum_{i=1}^{n} |w_i c_i| && \text{(12a)} \\ \mathrm{s.\,t.} \quad & c_i = \mathbf{h}_i^{\rm T}\mathbf{u}, \quad i = 1, \dots, n, && \text{(12b)} \\ & u_{\rm in} = \bar{u}_{\rm in}. && \text{(12c)} \end{aligned}$$

Here, $\alpha > 0$ is a constant parameter controlling weights of the two objectives: maximization of $u_{\rm out}$ and the minimization of the weighted ℓ_1-norm of $\mathbf{c}$. Problem (12) can also be converted to an LP problem. The dual problem of problem (12) is formulated as

$$\begin{aligned} \max \quad & \bar{u}_{\rm in}\lambda_{\rm in} && \text{(13a)} \\ \mathrm{s.\,t.} \quad & \sum_{i=1}^{n} y_i \mathbf{h}_i = \mathbf{p}_{\rm out} + \lambda_{\rm in}\mathbf{p}_{\rm in}, && \text{(13b)} \\ & \alpha w_i \geq |y_i|, \quad i = 1, \dots, n, && \text{(13c)} \end{aligned}$$

where $\lambda_{\rm in} \in \mathbb{R}$ and $\mathbf{y} \in \mathbb{R}^n$ are variables to be optimized. In a manner similar to problem (11), problem (13) is also analogous to the limit analysis problem. Here, $\mathbf{p}_{\rm out}$ corresponds to the fixed part of the external load and $\lambda_{\rm in}\mathbf{p}_{\rm in}$ corresponds to the proportionally increased part.

In contrast to problem (11), problem (13) is not necessarily feasible due to the presence of the fixed external load, $\mathbf{p}_{\rm out}$. To guarantee feasibility, α (or w_i's) should be sufficiently large. If problem (13) is



(a) $0.42 \leq \alpha \leq 0.63$. (b) $\alpha \geq 0.64$.

Figure 2: The mechanisms obtained from model 1 by solving problem (12). Filled circle: rotational hinge; dotted line: removed member.

infeasible, then it follows from the duality of LP that the objective value of problem (12) is not bounded below.

The conventional limit analysis problem with loads corresponding to input and output displacement is formulated as

$$\max \quad \lambda \tag{14a}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} y_i \mathbf{h}_i = \lambda(\mathbf{p}_{\text{in}} + \mathbf{p}_{\text{out}}), \tag{14b}$$

$$w_i \geq |y_i|, \quad i = 1, \dots, n, \tag{14c}$$

Note that the nonzero values in $\mathbf{p}_{\text{in}}$ and $\mathbf{p}_{\text{out}}$ are the specified values p_{in} and p_{out}, which are not necessarily 1. This problem is always feasible, because the load factor λ is multiplied to both $\mathbf{p}_{\text{in}}$ and $\mathbf{p}_{\text{out}}$. The dual of problem (14) is derived as

$$\min \quad \sum_{i=1}^{n} |w_i c_i| \tag{15a}$$

$$\text{s.t.} \quad c_i = \mathbf{h}_i^{\text{T}} \mathbf{u}, \quad i = 1, \dots, n, \tag{15b}$$

$$u_{\text{in}} + u_{\text{out}} = 1, \tag{15c}$$

which is analogous to problem (12).

7. Numerical examples

Mechanisms are found for various bar-joint models. All bars of initial ground structures are connected rigidly at joints. Intersecting diagonal bars are not connected with each other. The units are omitted for simple presentation of results.

7.1 Model 1

Problem (12) is solved for model 1 shown in Fig. 1. The size of square unit is 1×1. Parameter α in problem (12) is regarded as the penalty parameter for member extensions and member-end rotations. A mechanism is to be designed so that the output node B moves up when the input node A is moved downward. The prescribed input displacement is $\bar{u}_{\text{in}} = 0.30$.

This model is internally statically indeterminate, even if all member ends are replaced by pin-jointed; therefore, mechanisms cannot be generated without removing some members. To prevent too many members to be removed and to obtain a mechanism dominated by hinge rotations, we set weights w_i to 1.0 for member extension and to 0.0001 for hinge rotation. The mechanism obtained by solving problem (12) depends on the value of parameter α as shown in Fig. 2, where the filled circle and dotted line represent a rotational hinge and a removed member, respectively. Note that the objective function is not bounded below if $\alpha \leq 0.41$, because, with small value of α, increase of displacements does not lead

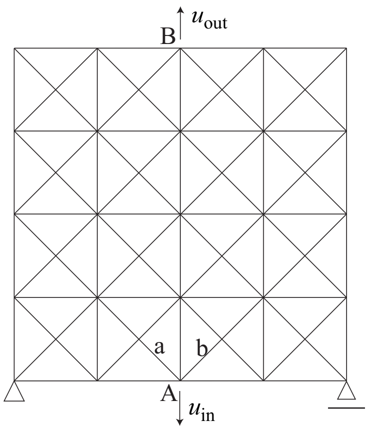
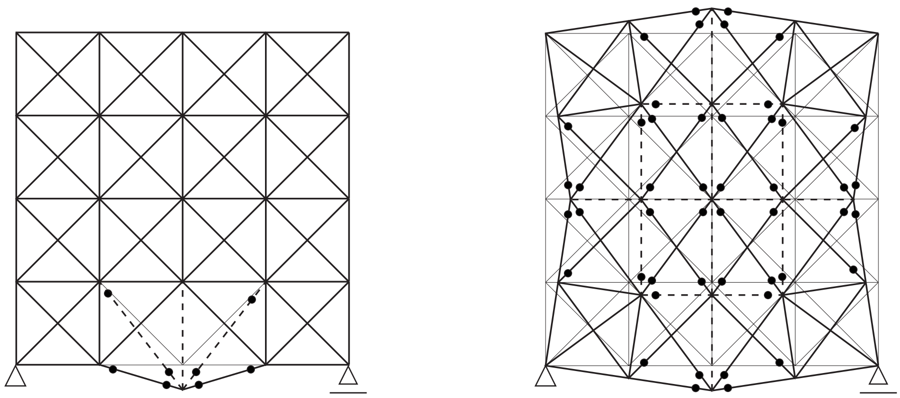


Figure 3: Model 2.



(a) A local mechanism.

(b) A global mechanism.

Figure 4: The mechanisms obtained from model 2 by solving problem (12). Filled circle: rotational hinge; dotted line: removed member.

to enough penalty in the objective function and thence the displacements can be arbitrary larger. In contrast, if α is large, only neighborhood of the input node moves as shown in Fig. 2(b). This way, the mechanism exhibiting desired deformation can be found by choosing appropriate values of parameters α and $w_1, \dots, w_n$.

7.2 Model 2

Mechanisms are next found for model 2 shown in Fig. 3, which has more members than model 1. The size of square unit is 1×1, and the same values as model 1 are given for $w_1, \dots, w_n$. The prescribed input displacement is $\bar{u}_{\mathrm{in}} = 0.3$. By solving problem (12), the local mechanism shown in Fig. 4(a) is found for $\alpha \geq 0.42$. The objective function is unbounded with $\alpha \leq 0.41$. For structures with many members, a local mechanism is often found, because local deformation leads to smaller penalty than global deformation.

A global mechanism can be obtained by assigning very large weights w_i for members that should not have axial deformation. By assigning $w_i = 10000.0$ for extension of members "a" and "b" in Fig. 3, we obtain the global mechanism shown in Fig. 4(b).

7.3 Model 3

The height of the lower units of model 1 is increased to 3 to obtain model 3 shown in Fig. 5. If $\bar{u}_{\mathrm{in}} = 0.20$ is assigned for problem (12), the output node B moves 0.60 as shown in Fig. 6(a).

We next solve the limit analysis problem (14) to obtain the yielding members and plastic hinges shown in Fig. 6(b) with the load factor $\lambda = 2.4$. Note that the nodal displacements of a mechanism are not explicitly obtained by solving problem (14) which involves only static variables. However, we can see

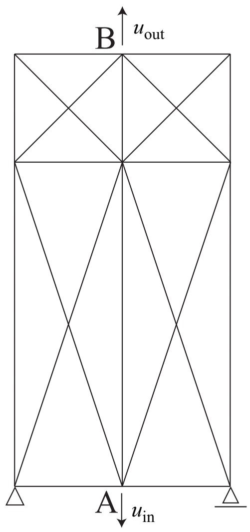
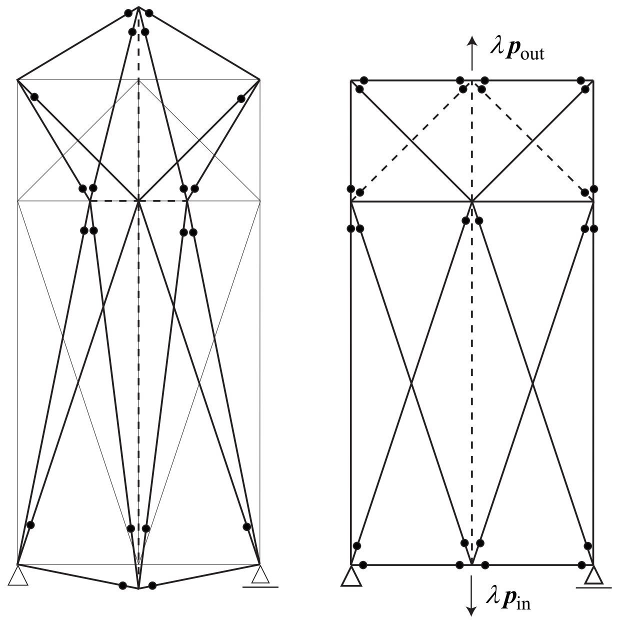


Figure 5: Model 3.



(a) The solution of problem (12). (b) The solution of problem (14).

Figure 6: The mechanisms obtained from model 3 by solving problems (12) and (14). Filled circle: rotational hinge; dotted line: removed member.

from Fig. 6(b) that only the output node moves in the mechanism, because the lower part cannot have displacements without axial deformation of an existing bar. In order to incorporate a parameter similar to α in problem (12), the load ratio between the input and output nodes is varied. It has been confirmed that the mechanism in Fig. 6(a) can be obtained with load ratio $0.82 \le p_{\text{out}}/p_{\text{in}} \le 0.88$.

7.4 Model 4

Finally, we generate a spatial mechanism from the initial frame in Fig. 7. This frame structure is in the XY-plane shown in Fig. 7, and the Z-axis is perpendicular to the XY-plane. The size of square unit is $1{\times}1$. The rotations around the X- and Y-axes are fixed at node 1, the displacements in the Y- and Z-directions are fixed at nodes 2 and 4, and displacements in the X- and Z-directions are fixed at nodes 3 and 5.

The output nodes 6, 7, 8, and 9 are expected to move upward, i.e., in the positive z-direction, as a result of input downward displacement at node 1. For this purpose, the limit analysis problem (14) is

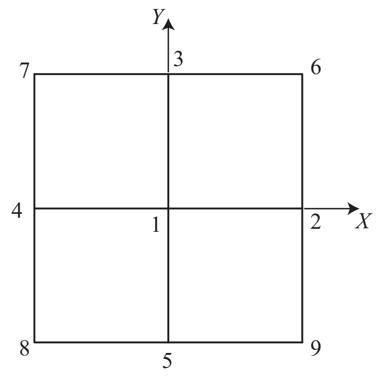
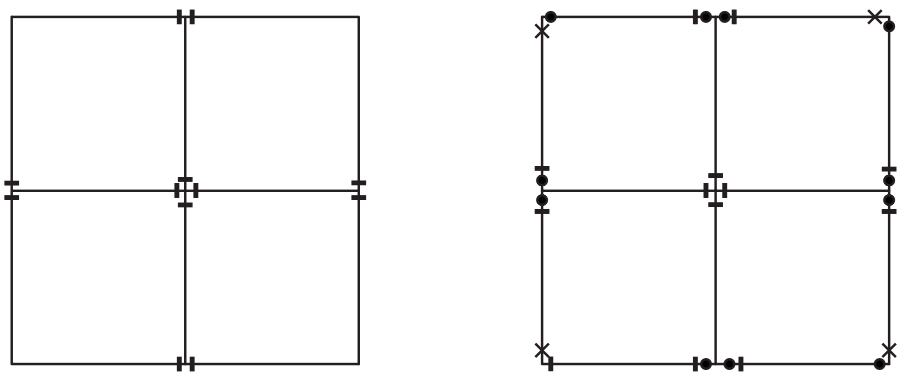


Figure 7: Model 4.



(a) A local mechanism. (b) A global mechanism.

Figure 8: The release conditions for Mode 4. Filled circle: rotational hinge around the z-axis; thick line: rotational hinge around the y-axis; "×": rotational hinge around the x-axis. (a) A local mechanism obtained by solving problem (14); (b) a global mechanism obtained after several trials of large-deformation analysis.

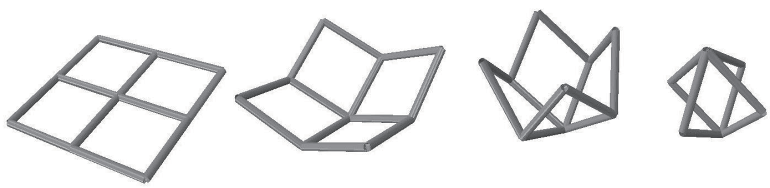


Figure 9: Deformation process of model 4.

solved to obtain the plastic hinges shown in Fig. 8(a). Here, we use $w_i = 10000.0$ for member extension and $w_i = 1.0$ for hinge rotation, and the obtained load factor is $\lambda = 2.4$. The initial local axes of each bar, denoted (x, y, z), are defined as follows. The x-axis is in the direction of a bar, the y-axis is on the XY-plane and perpendicular to the bar, and the z-axis is in the vertical direction. Note that these axes rotate in accordance with geometrically nonlinear deformation. In Fig. 8(a), a thick bar represents a rotational hinge around the y-axis.

Because the mechanism obtained by solving problem (14) is an infinitesimal mechanism, we carry out displacement-controlled large-deformation analysis with the incremental vertical displacement u at the output nodes and $-u$ at the input node. A finite mechanism shown in Fig. 8(b) is obtained by releasing the rotational constraints at member-ends that have nonzero bending moments. In Fig. 8(b), the filled circle is a rotational hinge around the z-axis, and "×" indicates release of rotation around the x-axis, i.e., release of torsion. The deformation process obtained by large-deformation analysis is shown in Fig. 9.

8. Conclusions

Various linear programming problems have been presented for generating a link mechanism, in which moderately small numbers of hinges and yielding members exist. These problems are regarded as problems for finding sparse solution of a system of linear equations. The related formulations in coding theory and machine learning have been reviewed.

It has been shown in numerical examples that the mechanism found as a solution of a linear programming problem strongly depends on values of the parameters, i.e., the weights for hinge rotation and bar extension and the penalty parameter for the sum of generalized displacements at member ends. In the context of the conventional plastic limit analysis problem, the penalty parameter is related to the load ratio between the input and output nodes. Finally, a three-dimensional mechanism has been generated to ensure applicability of the proposed approach to design of deployable structures.

9. References

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