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Topology optimization of masonry blocks with enhanced thermomechanical performances

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1. Abstract
Topology optimization is employed to define the geometry of the cross-section of any masonry block that minimizes its thermal transmittance, with the aim of maximizing the thermal insulation of masonry buildings. Constraints on the mechanical properties of the block are also prescribed. The effect of the design constraints on the optimal layout of the blocks is investigated. The thermal efficiency of the optimized units is also compared with that of commercially available blocks.

2. Keywords: block masonry, thermal conductivity, topology optimization.

3. Introduction
One of the primary goals in the field of modern construction is to design buildings with high thermal performances. This is dictated primarily by the need of reducing energy consumption in the exploitation of the building, because of the progressive decrease in available energy sources and the need of limiting the emissions of pollutants. This can be achieved e.g. acting on the exposure or the shape of the building, optimizing the technological systems, using large transparent walls to increase lighting and passive heating, as well as materials and construction elements that minimize the heat flow across the external walls of the building.

In this work, attention is focused on the optimal design of the shape of masonry (or concrete) blocks in order to minimize their thermal transmittance (i.e., to maximize their thermal resistance). The 'thermal transmittance' (usually denoted by U) of any wall is the heat flow per square meter, divided by the difference in temperature between the faces of the wall itself.

Assuming the heat flux to be uniform across the wall surfaces, topology optimization is employed to define the layout of the block section. Constraints on the block stiffness are also prescribed. The presence of holes of given shape in any prescribed position and other technological constraints can be easily embodied in the optimization procedure. The effect of the design constraints on the optimal layout of the blocks is investigated. The thermal efficiency of the optimized units is also compared with that of commercially available blocks.

Existing approaches dealing with the optimal design for thermal insulation resort to formulations of topology optimization that are based on the heat equation. Structural performances of the building envelope are generally neglected, see e.g. [1, 2]. The proposed approach consists in an original formulation that may handle the design of masonry blocks, dealing with the simultaneous optimization of the thermal and mechanical performances of the blocks. Non-trivial optimal block layouts can be achieved, depending on the design constraints. The inclusion of stiffness/strength requirements in the minimization of the thermal transmittance remarkably affect the optimal design along with its performances.

4. Governing equations
The fundamentals of the heat conduction problem in steady-state conditions over any plane domain are briefly recalled, in view of the formulation of the optimization scheme for thermal insulation [3]. The formulation is limited to the 2D case, according to the simplifying assumptions made in Sec. 3.

Let $\Omega \in R^2$ denote the domain, $\partial\Omega$ its regular boundary, and $\underline{\underline{k}}$ the second-order thermal conductivity tensor of the material in Ω. Under steady-state conditions, the unknown temperature field T over Ω is governed by the well known heat conduction equation:

$$\text{div}\ (\underline{\underline{k}}\ \text{grad}\ T) + b = 0, \tag{1}$$

where b is the heat energy generated per unit volume. In particular, if the material is isotropic, $\underline{\underline{k}} = k\underline{\underline{I}}$, being k the thermal conductivity and $\underline{\underline{I}}$ the identity tensor. Note that, for homogeneous isotropic domains, Eq. (1) reduces to $k\Delta T + b = 0$, being Δ the Laplace operator.

In general, a given temperature distribution T_0 can be prescribed over a certain part of $\partial\Omega$ (say, Γ_T), whereas a heat flux q can be prescribed over a different part Γ_f. Thus, the boundary conditions for Eq. (1) read

$$T = T_0 \text{ over } \Gamma_T, \quad -(\underline{\underline{k}} \text{ grad } T) \cdot \underline{n} = q \text{ over } \Gamma_f, \tag{2}$$

being $\underline{n}$ the unit outward normal vector to Γ_f. The latter condition includes, as a special case, a convective heat transfer through a part Γ_c of $\partial\Omega$. The heat flux per unit area q_c is proportional to the difference between the temperature of the body surface and the ambient temperature T_a, i.e.:

$$q_c = h_c(T - T_a), \tag{3}$$

being h_c the so-called convective heat transfer coefficient. Eq. (3) is commonly used in the design of building envelopes, to define the thermal transmittance of any wall (see below). The constant h_c globally takes into account the heat fluxes affecting the body surfaces under laminar flow conditions, see e.g. [4].

The weak formulation of the problem defined by eqs. (1) to (3) can be stated as (see e.g. [5]): find T such that $T\,|_{\Gamma_T}= T_0$ and

$$\begin{aligned}
&\int_\Omega \text{grad } w \cdot (\underline{\underline{k}} \text{ grad } T)\ dx = \\
&= \int_{\Gamma_T} w\ q_{react}(T_0)\ ds - \int_{\Gamma_f} w\ q\ ds - \int_{\Gamma_c} w\ h_c(T - T_a)\ ds + \int_\Omega w\ b\ dx \quad \forall w,
\end{aligned} \tag{4}$$

where w can be interpreted as a 'virtual' temperature field, and $q_{react}(T_0)$ stands for the 'reactive' heat flux acting on T_0. Eq. (4) can be re-written in compact form as:

$$a(w, T) + < w, h_c T >_{\Gamma_c} = < w, q_{react}(T_0) >_{\Gamma_T} - < w, q >_{\Gamma_f} + < w, h_c T_a >_{\Gamma_c} + < w, b > . \tag{5}$$

In analogy to the classical structural compliance, it is expedient to define a 'thermal compliance' $\mathcal{C}^T$ as [3, 6]:

$$\mathcal{C}^T(T) = < T, h_c T_{ai} >_{\Gamma_{ci}} \big(= a(T, T) + < T, h_c T >_{\Gamma_c}\big). \tag{6}$$

where $T_{ai} > 0$ is the ambient temperature at the boundary Γ_{ci}. For simplicity, the outdoor temperature $T_{ao} = 0$ at the boundary Γ_{co} is assumed to vanish. No heat source or heat flux are supposed to be prescribed within the domain or over its boundary.

As the heat diffusion depends on the magnitude of the convective flux across Γ_{ci}, i.e. $h_c(T_{ai} - T)$, the thermal compliance $\mathcal{C}^T(T)$ given by Eq. (6) can be spontaneously adopted as objective function to minimize or maximize the heat transfer through Ω. Minimizing $\mathcal{C}^T$, the optimal thermal conductor is found, meaning that the flux $h_c(T_{ai} - T)$ is maximized due to the minimization of the relevant surface temperature T on Γ_{ci}. Conversely, maximizing $\mathcal{C}^T$, the optimal thermal insulator is obtained, as the minimum heat conduction is sought.

Under the above assumptions, the thermal transmittance of any wall of the building separating the inner environment from the outer environment can be defined as

$$U = \frac{1}{|\Gamma_{ci}|} \int_{\Gamma_{ci}} h_c(T_{ai} - T) d\Gamma. \tag{7}$$

Maximizing $\mathcal{C}^T(T)$ amounts at minimizing U.

It is worth noting that, according to existing engineering codes, h_c takes different values at the surfaces in contact with the inner and 0the outer environment, denoted by h_{ci} and h_{co}, respectively. The values suggested in [4], for instance, are $h_{ci} = 7.7W/(m^2K)$ and $h_{co} = 25W/(m^2K)$. Only the value of h_{ci} is of interest for the numerical applications shown in Sec. 6, according to the definition of U, Eq. (7).

5. Optimal design for thermal insulation: problem formulation

Topology optimization by distribution of isotropic material is based on the adoption of suitable interpolation schemes [7, 8] to approximate the mechanical and physical properties of the material, herein the thermal conductivity k, depending on the design variable ρ. ρ can be interpreted as a non-dimensional material density, that ranges between 1 (full material) and 0 (void). Intermediate values of ρ correspond to a sort of porous material. The proposed procedure implements the so-called RAMP (Rational Approximation of Material Properties) model [9], that reads:

$$k(\rho) = k_0 + \frac{\rho}{1 + p(1-\rho)}(k_1 - k_0). \tag{8}$$

According to the EN 1745 standards [11], k_0 can be defined as an equivalent thermal conductivity of the voids (see also [10]). For $p = 0$, the RAMP model provides a linear interpolation of k between the extreme values. As p increases, the interpolation law strongly penalizes the intermediate material densities range and spontaneously leads to pure 0–1 designs. In the numerical applications presented hereafter, $p = 3$ is assumed. As shown in the Sec. 6, this choice allows pure black-and-white layouts to be obtained without the appearance of any gray regions.

The problem of finding the optimal material distribution over a design domain Ω that maximizes the heat conduction, i.e. $\mathcal{C}^T$, with prescribed convective boundary conditions and constraints on the elastic stiffness can be stated as follows:

$$\left\{ \begin{array}{ll} & \max_{\rho \in \mathcal{R}_0^+} \ \mathcal{C}^T = a(T,T) + < T, h_c T >_{\Gamma_c} \\ \text{s.t.} & \displaystyle\int_\Omega \text{grad } w \cdot (\underline{\underline{k}}\ (\rho) \text{ grad } T)\ dx = \int_{\Gamma_c} w\ h_c(T - T_a)\ ds \quad \forall w \in \mathcal{V} \\ & \mathcal{C}_h^S(\underline{u})/\mathcal{C}_{0h}^S \leq \alpha_h, \ \ h = 1 \dots n \\ & \displaystyle\int_\Omega C_{ijkl}(\rho) \varepsilon_{ij}(\underline{u}) \varepsilon_{kl}(\underline{v})\ d\Omega = \int_{\Gamma_t} \underline{t}_{0h} \cdot \underline{v}\ d\Gamma, \\ & 0 \leq \rho \leq 1 \end{array} \right. \tag{9}$$

In Eq.(9.2), the dependence of the thermal conduction tensor $\underline{\underline{k}}$ on the unknown design variable ρ, according to Eqn. (8), has been pointed out.

Eq. (9.3) represents a set of constraints on the overall stiffness, requiring that the structural elastic compliance $\mathcal{C}_h^S$, computed for any $h-$th load case, be greater than a prescribed threshold $\alpha_h \mathcal{C}_{0h}$, being $\mathcal{C}_{0h}$ the compliance evaluated for a homogeneous solid block without any voids and α_h a value selected to match any desired structural performance ($\alpha_h \geq 1$, $h = 1 \dots n$). These constraints can be replaced, or complemented, by other technological constraints. In some of the numerical applications shown in Sec. 6, for instance, a constraint on the maximum void fraction in the block, $\overline{V}_f$, is prescribed.

Finally, Eq. (9.4) represents, in weak form, the elastic equilibrium conditions for the body under prescribed boundary tractions $\underline{t}_{0h}$ corresponding to the $h-$th load case. In the l.h.s., the elastic coefficients C_{ijkl} can be expressed as a function of the design variable ρ according to a RAMP-like law similar to Eq.(8).

In a discrete formulation, assuming the domain to be subdivided into N finite elements, Eq.(9) reads:

$$\left\{ \begin{array}{ll} & \max_{x_i} \ \mathcal{C}^T = \sum_{i=1}^N \underline{\theta}_i^T \ (k(x_i) \underline{\underline{K}}_{t0,i} + \underline{\underline{H}}_{ti})\ \underline{\theta}_i \\ \text{s.t.} & (\underline{\underline{K}}_t(\underline{x}) + \underline{\underline{H}}_t)\ \underline{\theta} = \underline{F}_t \\ & \mathcal{C}_h^S(\underline{u})/\mathcal{C}_{0h}^S \leq \alpha_h, \ \ h = 1 \dots n \\ & \underline{\underline{K}}_s(\underline{x})\ \underline{U} = \underline{F}_s, \\ & 0 \leq x_i \leq 1, \end{array} \right. \tag{10}$$

where $\underline{x} = \{x_1, x_2, \dots, x_N\}^T$ is an array gathering the unknown densities of the N elements, $\underline{\theta}$ is the array of the nodal temperatures, $\underline{\underline{K}}_t$ is the matrix of the elemental conductivities, and $\underline{\underline{H}}_t$ is the heat transfer matrix, $\underline{U}$ is the array of the nodal displacements. The arrays $\underline{F}_t$ and $\underline{F}_s$ can be straightforwardly derived from Eq. (9). Subscript i denotes quantities pertinent to the $i-$th finite element.

In discrete form, the structural compliance $\mathcal{C}^S$ can be expressed as:

$$\mathcal{C}^S = \underline{U}^T\ \underline{\underline{K}}_s(\underline{x})\ \underline{U} = \sum_{i=1}^N \underline{U}_i^T\ E(x_i) \underline{\underline{K}}_{s0,i}\ \underline{U}_i, \tag{11}$$

being $\underline{\underline{K}}_s$ is the stiffness matrix of the entire body. Assuming the material to be isotropic, and adopting a RAMP-like interpolation for the elastic modulus E, the stiffness matrix of any element can be expressed as $E(x_i)\ \underline{\underline{K}}_{0,i}$, where $\underline{\underline{K}}_{0,i}$ refers to a unitary elastic modulus. E ranges between $E_0 = E(\rho = 0)$ and $E_1 = E(\rho = 1)$.

A gradient-based algorithm is used to solve problem (10). Suitable filtering schemes are adopted to avoid numerical instabilities and mesh dependency of the results.

6. Numerical applications

The problem formulated in Sec. 5 will be now applied to the derivation of the optimal material distribution for square section concrete blocks. The material properties employed in the applications are listed in Table 1: refer to Sec. 5 for the meaning of the symbols. The Poisson's ratio ν of the material is supposed to be unaffected by the material density. The values reported for $\rho = 1$ are typical of clay units. The value of the equivalent thermal conductivity of the voids, k_0, was taken from the EN 1745 standards [11]. The 'void' regions were assigned a negligible elasticity modulus E_0.

k_0	k_1	E_0	E_1	ν
$0.050\ W/m/K$	$0.700\ W/m/K$	$0.01\ N/mm^2$	$10000\ N/mm^2$	0.25

Table 1: Thermomechanical properties of the materials employed in the numerical applications.

Two self-equilibrated load cases are considered in the analyses: a uniform out-of-plane compression $t_{01} = 1N/m$, acting on the two sides of the block belonging to the surfaces of the wall, and a uniform in-plane compression $t_{02} = 1N/m$, acting on the other two faces of the block along the wall thickness. The compliance of the design domain in the absence of voids in the two load cases will be denoted by $\mathcal{C}_{01}$ and $\mathcal{C}_{02}$, respectively.

In all the figures shown hereafter, the vertical sides of the design domain correspond to the vertical surfaces of the wall, that is, the surfaces crossed by the heat flux; the horizontal sides of the design domain are parallel to the wall thickness.

The block side length is 0.30 m. The design domain is subdivided into approximately 65,000 elements. The elements along $\partial\Omega$, forming a strip 10 mm thick, have a fixed density $\rho = 1$. The thermal properties of the block are optimized giving the normalized in-plane structural compliance, $\alpha_2 = \mathcal{C}_2^S/\mathcal{C}_{02}^S$, a value of 2.5: as 1/2.5=0.4, this means that, along the wall thickness, approximately 40% of solid material will be found. The normalized out-of-plane compliance of the wall ($\alpha_1 = \mathcal{C}_1^S/\mathcal{C}_{01}^S$) is taken equal to 3 or 6. The optimal layout of the block obtained when the thermal and mechanical constraints are simultaneously taken into account is compared with that obtained if only the thermal performances or the mechanical performances of the block are maximized separately.

Fig. 1a shows the optimal material distribution obtained for $\alpha_1 = 3$ and Fig. 2a for $\alpha_1 = 6$. The corresponding values of U and V_f are reported in Table 2. As the required out-of-plane stiffness decreases, the thermal transmittance U decreases and the percentage of voids V_f increases. Note that the decrease in U is matched by staggered material layouts, which lengthen the path that heat must follow to cross the wall. At the lower value of α_2, a sort of thermal bridge arises (see Fig. 1a). At the higher value of α_2 (see Fig. 2a), the mechanically significant part of material localizes at the center of the block; the number of connections of the inner core with the vertical surfaces of the wall decreases, which contributes to further reduce the block transmittance.

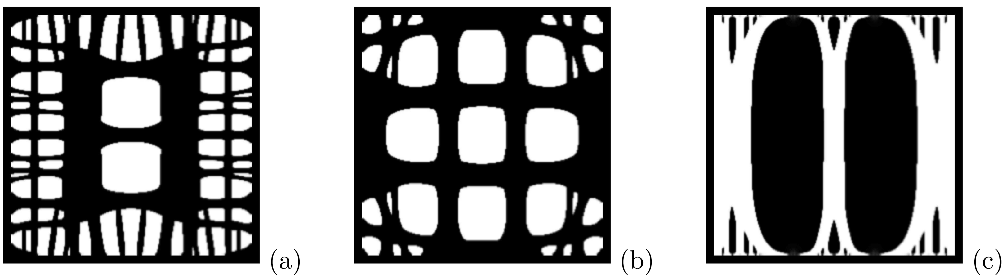

Figure 1: Square block. Optimal design for an in-plane compliance $\mathcal{C}_2^S = 2.5\mathcal{C}_{02}^S$ and an out-of-plane compliance $\mathcal{C}_1^S = 3\mathcal{C}_{01}^S$ (a); minimum transmittance design (b) and maximum stiffness design (c).

Figures 1b and 2b show which layouts would be obtained if the thermal transmittance U were minimized, without any constraint on the mechanical stiffness of the block. The percentage of voids, V_f, is the same that characterizes the layouts shown in Figs. 1a and 2a, respectively ($V_f = 42\%$ or 50% - see Table

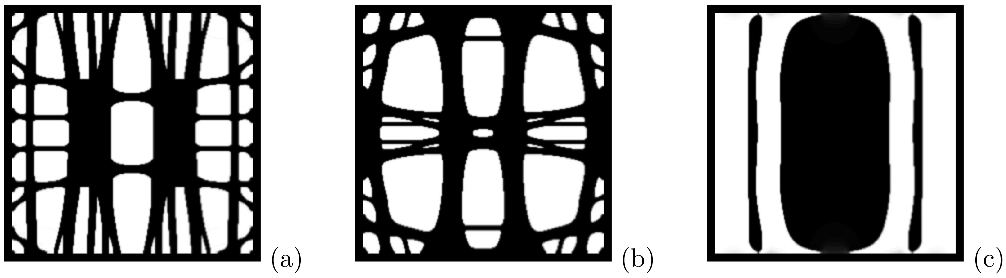

Figure 2: Square block. Optimal design for an in-plane compliance $\mathcal{C}_2^S = 2.5\mathcal{C}_{02}^S$ and an out-of-plane compliance $\mathcal{C}_1^S = 6\mathcal{C}_{01}^S$ (a); minimum transmittance design (b) and maximum stiffness design (c).

2). Conversely, Figs. 1c and 2c show the material layouts obtained if only the global mechanical compliance $\sum_{i=1}^{2} \alpha_i C_i^S$ is minimized, neglecting the thermal performances. The corresponding values of U are reported in Table 2. Apparently, maximizing only the thermal or the mechanical properties of the block leads to unfeasible designs: a bulky inner core, with negligible out-of-plane stiffness, is obtained in the former case, whereas many thermal bridges, leading to poor thermal insulation properties, are obtained in the latter case. The formulation proposed in the present work tries to take all the thermomechanical properties of the block into account, and leads to values of U that are a good compromise between those obtained with the mono-objective formulations.

Finally, it is worthwhile to compare the thermal performances obtained with the proposed optimization procedure with those of blocks with a 'classical' pattern of holes. The square block shown in Fig. 3a is considered, characterized by a regular staggered pattern of rectangular holes. The structural compliances of the block along the two directions are computed, $\mathcal{C}_1^S$ and $\mathcal{C}_2^S$, and the thermal transmittance of the block is evaluated as well (see the row of Table 2 corresponding to Fig. 3a). The topology optimization procedure is then applied to identify the optimal material layout for a square block with the same structural compliances, which is shown in Fig. 3b. As reported in the corresponding row of Table 2, a decrease in transmittance of about 5% is obtained using the optimized block. This is due, on the one hand, to the higher percentage of voids compared to the 'classical' one; on the other hand, to the presence of a single web along the wall thickness in the central part of the block.

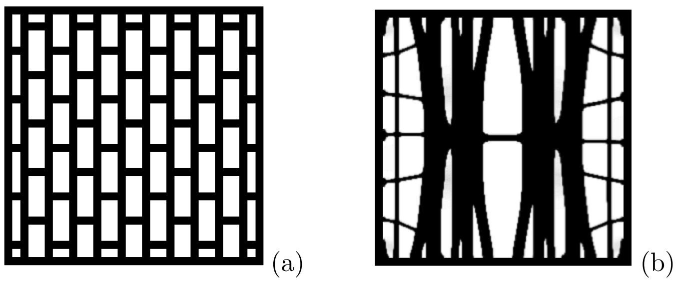

Figure 3: Square block. Conventional staggered design and optimal layout achieved for the same values of non-dimensional compliances α_1 and α_2 (b).

7. Concluding remarks

The layout of hollow masonry units that minimizes their thermal transmittance was sought using a topology optimization approach (Sec. 5). The potentialities of the procedure have been illustrated through numerical applications (Sec. 6). The optimal material distributions obtained can be exploited to achieve the pre-design of new and non-standard types of blocks. In principle, the optimization procedure allows technological constraints to be taken into account, directly or indirectly.

As the heat flux is assumed to be uniform across the height of the wall, a two-dimensional problem is dealt with. This is a quite rough simplification, as the presence of concrete layers at the top of the blocks or strip bed joints cannot be taken into account: the problem should be studied as 3D, as pointed out in [10]. This extension will be dealt with in the continuation of the research.

Fig.	Problem	$U[W/m^2/K]$	$\mathcal{C}_1/\mathcal{C}_{01}$	$\mathcal{C}_2/\mathcal{C}_{02}$	$V_f[\%]$
1(a)	min U s.t. $\mathcal{C}_i/\mathcal{C}_{0i} \leq \alpha_i$	0.836	3.0	2.5	42
1(b)	min U s.t. $V_f \leq \overline{V}_f$	0.482	444.9	12.7	
1(c)	min $\sum \alpha_i C_i$ s.t. $V_f \leq \overline{V}_f$	0.889	2.7	2.6	
2(a)	min U s.t. $\mathcal{C}_i/\mathcal{C}_{0i} \leq \alpha_i$	0.624	6.0	2.5	50
2(b)	min U s.t. $V_f \leq \overline{V}_f$	0.445	455.0	9.4	
2(c)	min $\sum \alpha_i C_i$ s.t. $V_f \leq \overline{V}_f$	0.754	4.1	2.9	
3(a)	conventional design	0.586	11.6	2.1	46
3(b)	min U s.t. $\mathcal{C}_i/\mathcal{C}_{0i} \leq \alpha_i$	0.556			49

Table 2: Square blocks. Comparison of the optimal designs in terms of thermal transmittance U, non-dimensional out-of-plane compliance α_1, non-dimensional in-plane compliance α_2, and percentage of voids V_f.

8. References

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