Isogeometric Layout Optimization of Shell Structures Using Trimmed NURBS Surfaces

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1. Abstract

A new method is presented which allows to determining the optimal segmentation of shell structures built from precast patches. The geometry is described by Non-Uniform Rational B-Spline (NURBS) surfaces and is integrated into the finite element method applying the isogeometric analysis technology and most sophisticated own implementations of isogeometric shell elements. The goal of the method is to find the segmentation, for which the stress resultants in the boundaries of the segments are minimal. The method consists of three steps. In the first step, the stress resultants of the shell structure are computed. This can be done with an isogeometric shell analysis considering one or more load cases. In the second step the parameter space of the shell structure is roughly divided into a certain number of segments using NURBS curves. This step is equivalent to the action of splitting/trimming the NURBS surface, which describes the shell model. In the last step the NURBS curves in the parameter space of the shell model are adapted such that the stress resultants for these curves get minimized. The result of the method is the description of the original shell model with optimal trimmed NURBS surfaces. Thus the result of the method is again a Computer Aided Design (CAD) model and can be used directly for further applications. The method can be used for the segmentation of shell structures which should be built out of precast concrete components. The procedure is fully integrated into the CAD program Rhino3D which is an important issue for practical application and will be demonstrated by the contribution as well. The success of the method will be demonstrated by illustrative benchmark examples. To summarize, the presented methodology combines the isogeometric analysis of shells, latest developments for the trimming of NURBS-surfaces, CAD-FEM integration, optimization of layout of shell segments made from fiber reinforced concrete.

2. Keywords:
Isogeometric Analysis, Trimmed NURBS Surfaces, Integration CAD and CAE, Layout Optimization

3. Introduction

For the design of shell structures made of precast concrete components a method is needed able to determine the optimal segmentation of shells. Thereby the optimal segmentation is defined such that the maximum stress in the boundaries of the segments is minimal. There are certainly many ways to solve such a problem. For example one could use topology optimization techniques [3]. The drawback of such techniques is to retrieve the optimization result from a finite element mesh back to a Computer Aided Design (CAD) model. This process can be very tedious. In order to circumvent this procedure a recently developed finite element approach named isogeometric analysis [6] is used for the optimization. Isogeometric analysis uses for the analysis the same basis function as the CAD model, which makes the meshing of the CAD model unnecessary. Usually Non-Uniform Rational B-Splines (NURBS) are taken because they are most prevalent in CAD systems. For this text NURBS are assumed. Isogeometric analysis allows it to apply the result of the proposed optimization technique directly to the original CAD model just by using NURBS trimming operations. Thus the presented optimization can be embedded perfectly into the design process.
The presented optimization method adjusts the boundary curves of trimmed surfaces, which form the shell structure. In other words the initial layout of the trimming curves, defined by the user, is adjusted. Thus the name layout optimization is used. The adjustment of the trimming curves is done with a gradient based optimization algorithm.
The layout optimization technique is implemented in the in house finite element software CARAT++,

which is integrated in the NURBS based CAD program Rhino3D. Latter is used as pre- and post processing for the optimization. All examples and figures were done with this software package.

In the following section, the isogeometric concept including NURBS is described. Next, the layout optimization is introduced. Further, some examples are presented to illustrate the use of the proposed optimization technique.

4. Isogeometric Analysis and NURBS

Isogeometric Analysis is a recent developed technology in the field of finite element analysis. The name for this technology was introduced by T. Hughes [6]. In isogeometric analysis the functions from the geometry description are used as basis functions for the analysis. Thus the design and analysis work on the same geometrical model. This means the analysis can take the geometry directly from a CAD model without the necessity of employing meshing procedures. For the design usually NURBS (Non-Uniform Rational B-Spline) are taken because they allow great geometrical flexibility with a high continuity (smoothness) [12]. NURBS are mathematically defined curves, surfaces and bodies which can be used to model a large variety of different shapes. The term NURBS is the short form of Non-Uniform Rational B-Spline and indicates that NURBS are a generalization of B-Splines. Therefore a short introduction to B-Spline is given first.

4.1. B-Splines

A B-Spline is a non-interpolating, parametric curve, which consists of piecewise defined polynomials. It is defined by a set of control points $\mathbf{P}_i$, $i = 1, ..., n$, a polynomial degree p and a knot vector $\mathbf{\Xi} = [\xi_1, \xi_2, ...\xi_{n+p+1}]$. Latter is a vector of parametric coordinates ξ_i in non-descending order which specifies the parametric domain and divides the B-Spline into sections. The knot vector is called open if the first and the last knot have a multiplicity of $p + 1$. Open knot vectors interpolate the first and the last control point of a B-Spline and are standard in CAD applications and are assumed for this text. B-Spline curves of degree p are computed by the linear combination of control points and the corresponding basis functions. The formula for B-Spline curves is

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\mathbf{P}_i \tag{1}$$

An example of a cubic B-Spline with an open knot vector is shown in figure 1. Due to the open knot vector the first and last control point (P_1 and P_7) are interpolated and the curve is tangential to the control point polygon at its start and end.
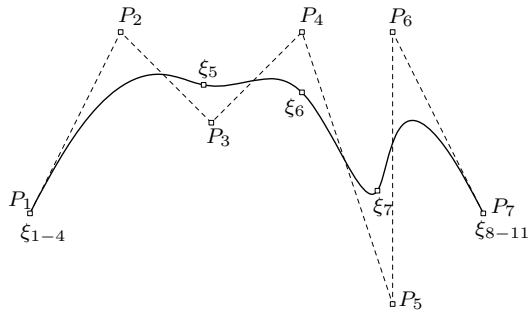

Figure 1: Cubic B-Spline with the knot vector $\mathbf{\Xi} = [0, 0, 0, 0, 0.25, 0.5, 0.75, 1, 1, 1, 1]$. The dashed lines represent the control point polygon. The small squares are the knots and divide the curve into segments.

4.2. Basis functions

The basis functions of B-Spline can be computed by the Cox-deBoor recursion formula [12] beginning with the order $p = 0$:

$$N_{i,0}(\xi) = \begin{cases} 1, & \xi_i \leqslant \xi < \xi_{i+1} \\ 0, & otherwise \end{cases} \tag{2}$$

and for $p \geqslant 1$ it is

2

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \tag{3}$$

The basis functions are C^∞ continuous inside the knot spans, which is the interval between two consecutive knots, and C^{p-1} at the knots. The local support for each basis function is the interval $[\xi_i, \xi_{i+p+1}]$. The figure 2 shows the basis functions of the B-Spline curve in figure 1.
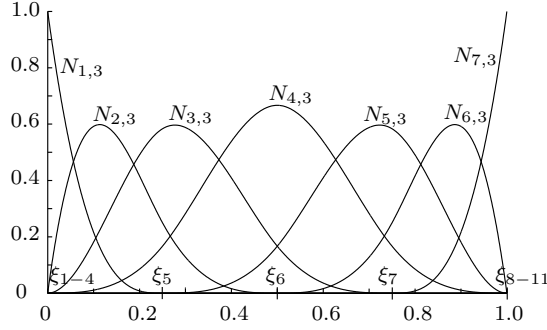

Figure 2: Cubic B-Spline basis functions of the curve in figure 1 with the open knot vector $\mathbf{\Xi} = [0, 0, 0, 0, 0.25, 0.5, 0.75, 1, 1, 1, 1]$

4.3. B-Spline surfaces

B-Spline surfaces are computed by the tensor product of B-Spline basis functions in two parametric dimensions ξ and η. They are defined by a net of $n \times m$ control points, two knot vectors $\mathbf{\Xi}$ and $\mathbf{H}$, two polynomial degrees p and q and the corresponding basis functions $N_{i,p}(\xi)$ and $M_{j,q}(\eta)$. The formula for a B-Spline surface is

$$\mathbf{S}(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(\xi) M_{j,q}(\eta) \mathbf{P}_{i,j} \tag{4}$$

An example of a quadratic B-Spline surface with the open knot vectors $\mathbf{\Xi} = \mathbf{H} = [0, 0, 0, 0.5, 1, 1, 1]$ is shown in figure 3. The surface on the right shows in addition the control point polygon net.
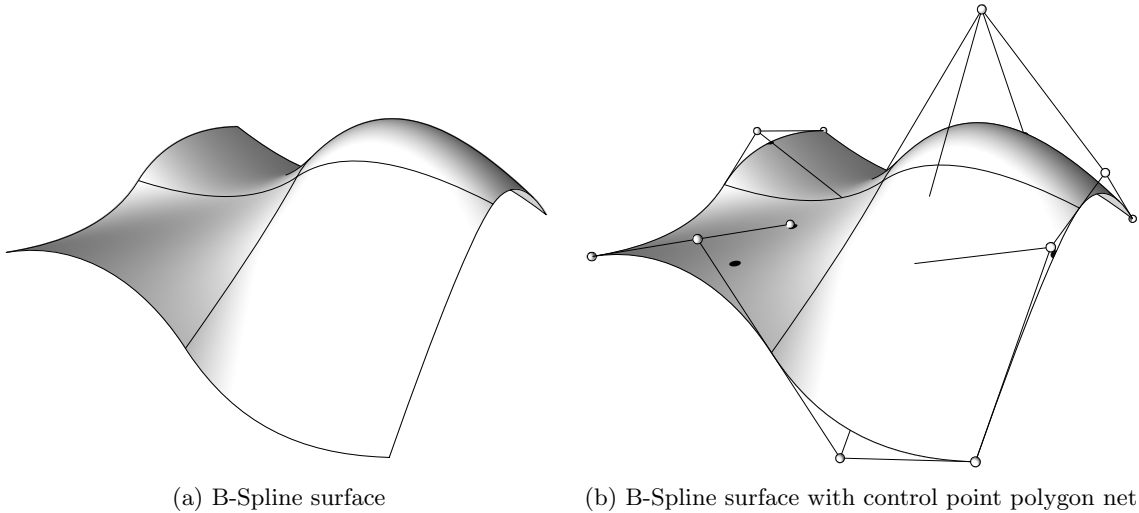

(a) B-Spline surface

(b) B-Spline surface with control point polygon net

Figure 3: Quadratic B-Spline surfaces with the open knot vectors $\mathbf{\Xi} = \mathbf{H} = [0, 0, 0, 0.5, 1, 1, 1]$. The straight black lines in the right picture represent the control point polygon net. Due to the open knot vectors the control points at the corners are interpolated.

4.4. Non-Uniform Rational B-Spline (NURBS)

NURBS are non-uniform rational B-Spline, which means they have non-uniform knot vectors and rational

basis functions. NURBS have in addition to their control point coordinates (x_i, y_i, z_i) an individual weight w_i, which leads to rational basis functions. These are necessary to represent circles and cylinders exactly. NURBS curves are defined as

$$\mathbf{C}(\xi) = \frac{\sum_{i=1}^{n} N_{i,p}(\xi) w_i \mathbf{P}_i}{\sum_{\hat{i}=1}^{n} N_{\hat{i},p}(\xi) w_{\hat{i}}} = \sum_{i=1}^{n} R_{i,p}(\xi) \mathbf{P}_i \tag{5}$$

whereas NURBS surfaces are defined as

$$\mathbf{S}(\xi, \eta) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(\xi) M_{j,q}(\eta) w_{i,j} \mathbf{P}_{i,j}}{\sum_{\hat{i}=1}^{n} \sum_{\hat{j}=1}^{m} N_{\hat{i},p}(\xi) M_{\hat{j},q}(\eta) w_{\hat{i},\hat{j}}} = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,p}(\xi) Q_{j,q}(\eta) \mathbf{P}_{i,j} \tag{6}$$

In the case all weights are equal to one NURBS can be represented by B-Splines. Thus B-Splines geometries are a subset of NURBS geometries.

4.5. Geometry refinement

There are two ways of refining B-Spline and NURBS geometries. They are called degree elevation and knot insertion. Both refinement techniques enhance the number of control points and therefore the capability of representing more shapes. The refinement does not change the shape of the geometry. For more details see [12].

4.6. NURBS based shell element formulations

For the proposed layout optimization technique a surface geometry model with a high continuity is required. The high continuity is needed to determine the gradients for the optimization algorithm. The gradients need to be computed on fields which depend on the first, second or third derivative of the displacement field. The required field depends on the stress resultant, which needs to be optimized along the trimming curves. Thus the analysis is done on a NURBS surface with a high enough continuity, applying the isogeometric analysis technology. For the description of the kinematics the Kirchhoff-Love or Reissner-Mindlin shell theory can be used. For both theories NURBS based shell formulations exist. For the Kirchhoff-Love theory J. Kiendl et al. [8] and for the Reissner-Mindlin theory W. Dornisch et al. [5] developed a NURBS based shell formulation. The application of the presented layout optimization is independent of shell formulation and can be used for both theories. It only requires a smooth stress resultant field, which can be achieved by a high enough continuity for the displacement field. The minimal requirements on the continuity of the NURBS surface model differ depending on the applied shell theory. In general the Kirchhoff-Love theory needs a by one increased continuity compared to the one needed for the Reissner-Mindlin theory.

For this text the NURBS based Kirchhoff-Love element formulation is used. A detailed description for this element can be found in [9]. An example of an isogeometric shell analysis with the NURBS based Kirchhoff-Love shell element is shown in figure 4, which represents the Scordelis-Lo roof from the shell obstacle course [2].

4.7. Scordelis-Lo roof

The Scordelis-Lo roof is a part of a cylindrical shell supported at its ends and free edges on the left and right side. A problem description is shown in figure 4a. The roof is loaded by uniform gravity load. The reference solution for this example is the vertical displacement at the point A. For the analysis of this example a NURBS surface with polynomial degrees of four in both directions ξ and η and twenty elements per edge are used. An element is equivalent to a knot span. This results in a vertical displacement at the point A of u=0.3006 which is the converged solution for the used element type [8] and similar to the reference solution in [2].

As mentioned before, for the layout optimization a stress resultant field is required. An example of such a field is shown in figure 4b. For NURBS based shell formulations one can compute the stress resultants for every parameter position (ξ, η). Thus it is possible to define curves in the parameter space for evaluating the stress resultants along these curves.

The figure 5 shows an example of stress resultant evaluations along curves defined in the parameter space. The figure shows the normal force n^{11} field for the Scordelis-Lo roof including evaluations along curves defined in the parameter space (dashed lines). In order to evaluate the stress resultant along these parameter curves, they are divided into a specific number of segments using the knot insertion refinement technique. For this example the curves are divided into one hundred segments and for each knot span two
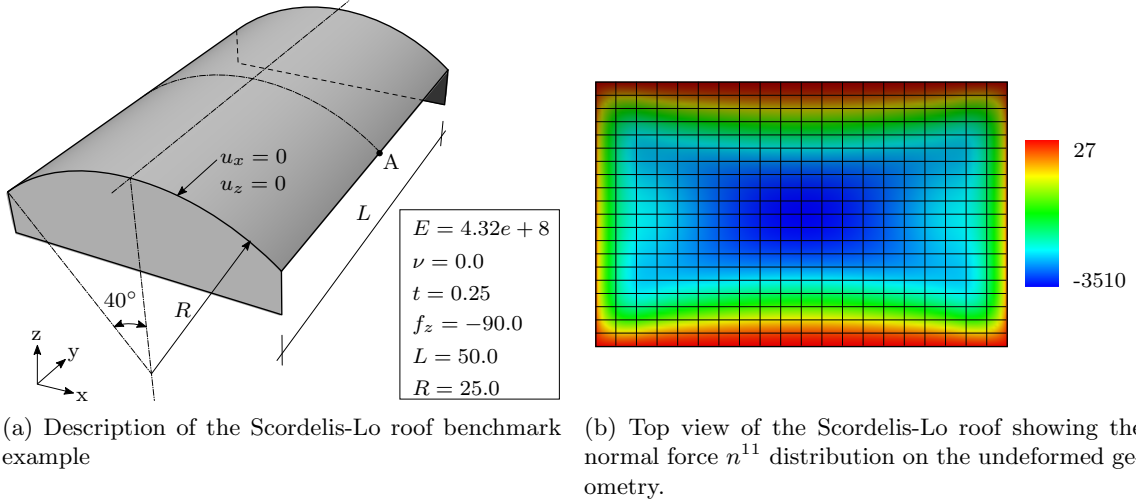
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(a) Description of the Scordelis-Lo roof benchmark example

(b) Top view of the Scordelis-Lo roof showing the normal force n^{11} distribution on the undeformed geometry.

Figure 4: Shell benchmark example: Scordelis-Lo roof with its stress resultant field of n^{11}

evaluations at its Gauss points are performed and visualized by straight lines, perpendicular to the mid surface of the shell. This technique allows the evaluation of stress resultants along paths on the surface defined by arbitrary curves in the parameter space.
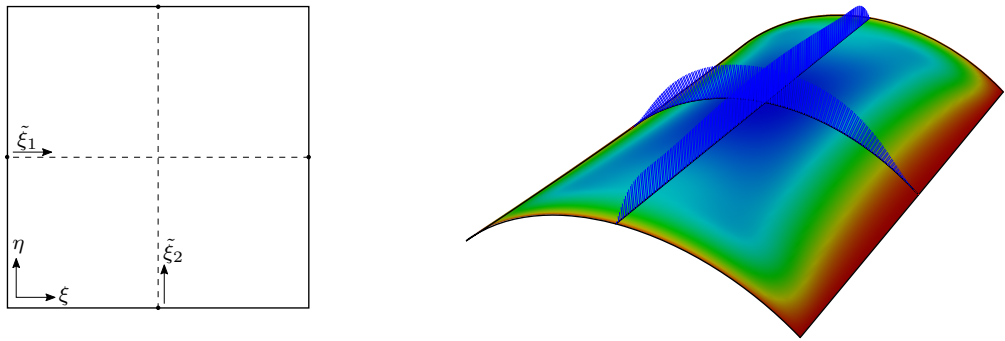
(a) Parameter space of the NURBS surface which describes the Scordelis-Lo roof geometry and two additional curves parameterized with $\tilde{\xi}_1$ and $\tilde{\xi}_2$

(b) Isometric view of the normal force n^{11} distribution with some evaluations along curves defined in the parameter space

Figure 5: Normal force n^{11} distribution of the Scordelis-Lo roof with some evaluations along two parameter curves

In the case the curves in the parameter space describe a closed loop in counter clockwise direction, they define a trimmed NURBS surfaces [14]. For the layout optimization closed boundary loops are adjusted and thus the results can be used directly in a CAD program using trimmed surface. In the next section trimmed surfaces are explained briefly .

4.8. Trimmed NURBS

Trimmed NURBS surfaces are standard in CAD applications and are even more used than untrimmed NURBS surfaces. They are defined by closed loops in the parameter space. An example of a trimmed surface is shown in figure 6. The curves $(A' - B', B' - C', C' - D', D' - A')$ in the parameter space of figure 6a define the outer boundary of the trimmed surface. The area inside the loop (curves have counter clockwise direction) belongs to the trimmed surface whereas the remaining parameter space will be hidden for the visualization. For visualizing trimmed surfaces the boundary curves in the parameter space are mapped to the geometry space and the inner parts are visualized. For the example in figure 6 the parameter curves $(A' - B', B' - C', C' - D', D' - A')$ are mapped to the trimming curves $(A - B, B - C, C - D, D - A)$ and define the trimming curves for the trimmed surface. This operation is standard in

CAD systems and is used for the visualization of the layout optimization results. The figure 6b shows the trimmed NURBS surface defined by the parameter space in figure 6a. A detailed description of trimmed NURBS surfaces can be found in [14]
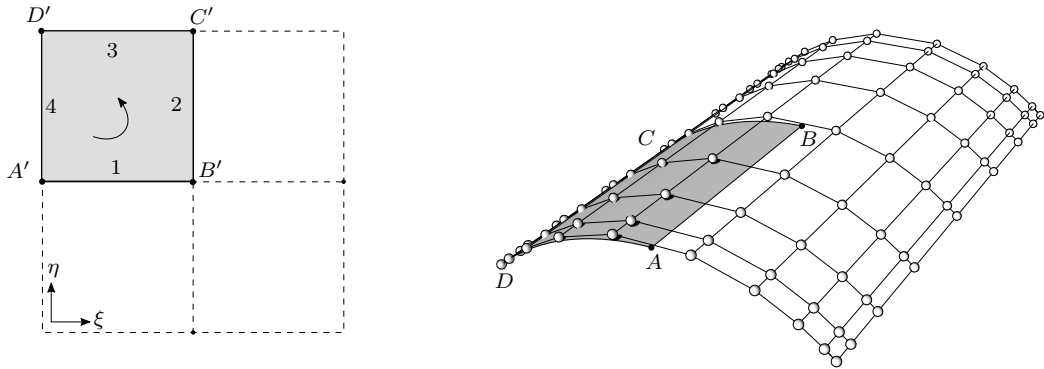
(a) Parameter space of a trimmed NURBS surface: The closed loop defines the boundaries of the trimmed surface (grey area)

(b) Isometric view of the trimmed NURBS surface with its control point polygon net

Figure 6: Trimmed NURBS surface with its parameter space and boundaries

5. Layout Optimization

Optimization problems can be expressed in a general form as follows:

$$
\begin{aligned}
\text{minimize} \quad & f(\mathbf{x}); && \mathbf{x} \in \mathbb{R}^n; \ \mathbf{x}_l \leq \mathbf{x} \leq \mathbf{x}_u \\
\text{such that} \quad & g_k(\mathbf{x}) \leq 0; && k = 1, ..., p \\
& h_j(\mathbf{x}) = 0; && j = 1, ..., q
\end{aligned}
$$

where $f(\mathbf{x})$ is the objective function, $g_k(\mathbf{x})$ the inequality and $h_j(\mathbf{x})$ the equality constraints. All of them depend on the optimization variables $x_i, i = 1, ..., n$, collected in the vector $\mathbf{x}$. The objective function can be for example the total volume [10], the strain energy [7] or the maximum stress value [1]. During the optimization the objective function $f(\mathbf{x})$ is minimized by modifying the design variables x_i. The design variables x_i can be related to any kind of variable which has an influence on the objective function such as material properties [3], the shape or the topology of a structure. For the layout optimization the control points of the boundary curves in the parameter space of trimmed NURBS surface are used.

For the optimal solution the design variables x_i have to satisfy the constraints. These constraints are expressed by the lower and upper bound $\mathbf{x}_l$ and $\mathbf{x}_u$. For this text it is assumed that the design variables need to be inside the limits of the NURBS parameter space. Apart from that there are no further restrictions for the optimization e.g. limit surface area or weight for the patches, thus the inequality constraints can be neglected.

For searching the optimal solution an adaptive three point line search is used. For the computation of the sensitivities a forward finite difference approach is used, which means for computing the sensitivities the coordinates of the boundary curves in the parameter space have to be changed slightly.

The figure 7 shows the design variables of a possible layout optimization. The design variables are marked with double arrows. By changing the coordinates of the control points the shape of the trimmed surfaces are changed. The parameter space of patch 1 and 2 coincide and the common boundary curve is coupled. During the layout optimization the design variables are adjusted such that the maximum stress resultant along the boundary curve is minimal. For minimizing the maximum stress resultant a Kreisselmeier-Steinhauser function is used, which is explained in the next section.

5.1. Kreisselmeier-Steinhauser function

In order to minimize one or a combination of different stress resultants along boundary or parameter curves respectively it is necessary to express the maximum value by a continuos function, that integrates the stress resultants along the boundary curve. This means that the discrete stress resultants values along a curve contribute to one value in portion. Two function, which could be used for this situation are the
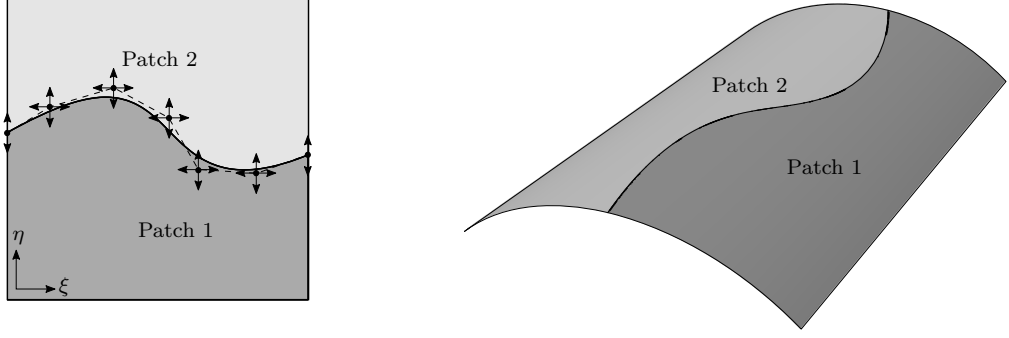
(a) Design variables (double arrows) are adjusted by the layout optimization. The boundary curves describes the trimming curve of the two patches.

(b) Isometric view of the Scordelis-Lo roof described by two trimmed NURBS surfaces

Figure 7: Initial configuration for a layout optimization

p-norm and the Kreisselmeier-Steinhauser (KS) function. The paper [13] shows that the KS function is more adaptive for optimization problems. Therefore the KS function is used for the layout optimization. The Kreisselmeier-Steinhauser functions for stress resultants is defined as follows:

$$f_{KS}(\sigma_i, \bar{\sigma}, \rho) = \frac{1}{\rho} ln \left(\sum_{i=1}^{N} exp \left(\rho \frac{\sigma_i}{\bar{\sigma}} \right) \right) \tag{7}$$

where $\sum_{i=1}^{N}$ represents the sum over the stress resultants evaluation points along the parameter boundaries and σ_i are arbitrary stress resultant like normal force, moments or a combination of it at the evaluation points. These stress resultants are normalized by a maximum allowable stress resultant $\bar{\sigma}$. The parameter ρ specifies the the relevance of the largest stress resultant among all σ_i. The higher the parameter ρ is chosen the more the maximal stress resultant dominates the KS function. A good value for ρ is between 1 and 100. For more details see [11]. For the proposed layout optimization the magnitude of the stress resultant needs to be optimized, thus the KS function has to be rewritten as:

$$f_{KS}(\sigma_i, \bar{\sigma}, \rho) = \frac{1}{\rho} ln \left(\sum_{i=1}^{N} exp \left(\rho \left| \frac{\sigma_i}{\bar{\sigma}} \right| \right) \right) \tag{8}$$

The reference stress resultant $\bar{\sigma}$ needs to be set reasonable because if the value of $\bar{\sigma}$ is too small compared to the current stresses resultant σ_i the value of the KS function tends to infinity whereas if the value is too small all stets of different stress resultants σ_i give the same KS function value. So the maximum stress resultants and the reference stress resultant should be in a similar range. Either the start value is chosen appropriate or the reference value has to be updated during the optimization.

6. Benchmark Examples

In the following section two examples are presented to illustrate the use of the proposed optimization technique. The first example deals with a double clamped beam modeled with a NURBS surface and the second example is based on the Scordelis-Lo roof explained in the previous section.

6.1. Double clamped beam

This example consists of a beam, modeled with a NURBS surface, clamped at its ends and loaded by a line load in the middle of the structure. The goal of the layout optimization is to find the optimal segmentation with the condition that the maximum moment around the y-axis along the trimming curve is minimal. The distribution of the moment around the y-axis is shown in figure 8b. It shows a linear distribution of the moment, which is correct. As an initial configuration an arbitrary chosen parameter/trimming curve is used.

For this optimization the boundary curve in the parameter space is a quadratic B-Spline curve with two knot spans. For the KS function the parameter ρ is set to five and the reference stress resultant $\bar{\sigma}$

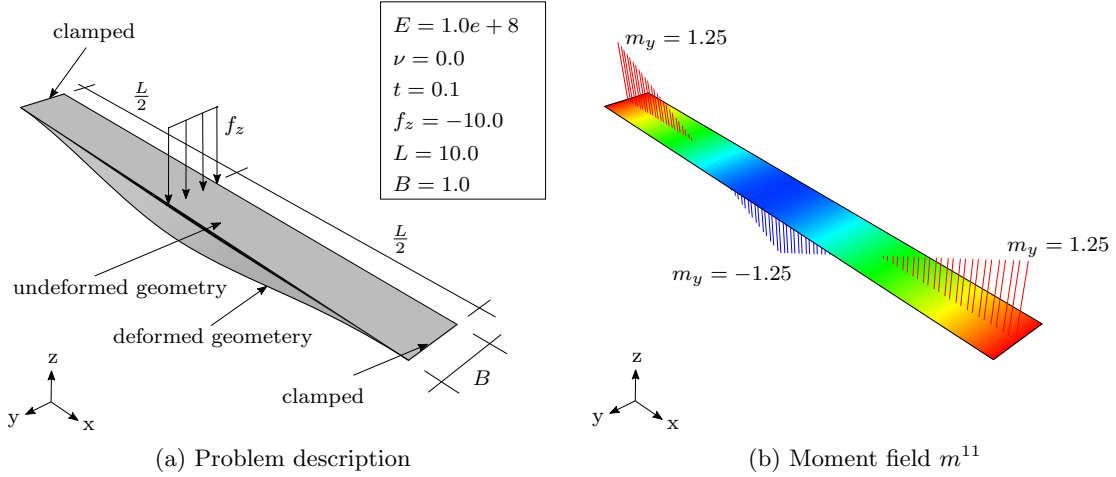(a) Problem description   (b) Moment field m^{11}

Figure 8: Problem description of the double clamped beam

is equal to one. For the optimization a stress resultant field computed by a NURBS surface with the polynomial degrees $p = q = 3$ and eight elements per direction is used. In addition ten evaluation points along the boundary curve are used for the KS function. The figure 9 shows how the optimum is found in 20 optimization steps, whereby the required number of steps depends on the maximal step size.
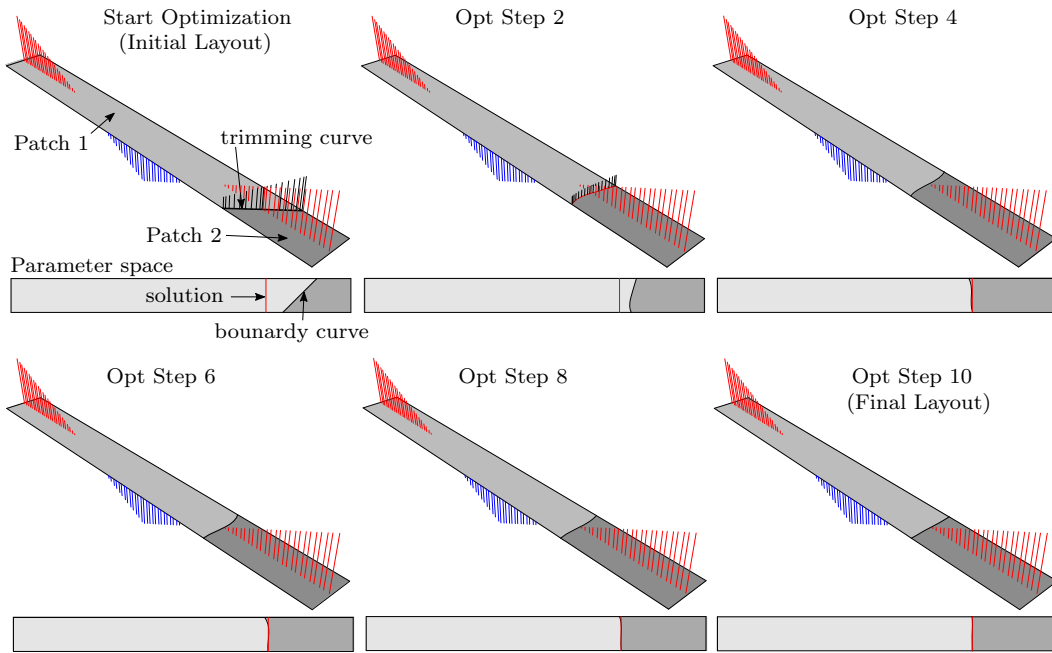

Figure 9: Layout optimization for the double clamped beam (figure 8) with a quadratic boundary curve and two knot spans

6.1. Scordelis-Lo roof

The second benchmark examples is the Scordelis-Lo roof, a shell structure already used mention before. For this example the goal is to minimize the moment m^{12} along the predefined boundary curve, which is shown in figure 10. For the layout optimization the boundary curve is of degree four and has five knot spans. For the KS function the parameter ρ is set to five and the reference stress resultant value is $\bar{\sigma}$ equal to fifty. The stress resultant field was computed with the same stetting as in section 4.7.
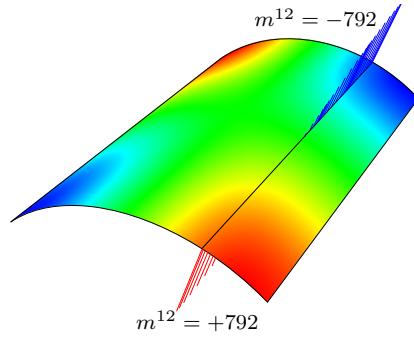
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$m^{12} = -792$

$m^{12} = +792$

Figure 10: Initial configuration for the layout optimization

In figure 11 the optimization steps are shown. At the beginning the ends of the boundary curve are adjusted and than slowly the rest. After 20 optimization steps the analytically solution is found, which is a straight line in the middle of the shell structure. The final layout splits the shell into two equal segments.
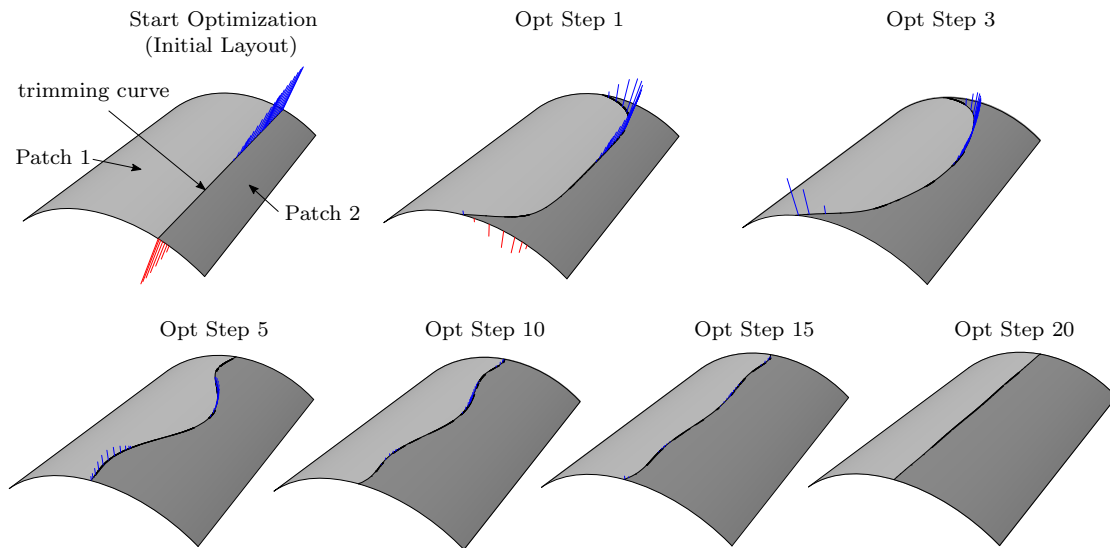

Figure 11: Optimization quadratic with two knotspans

In figure 12 a graph is plotted which shows the KS function values over the optimization steps. The graph clearly shows how the KS function gets minimized and converges.
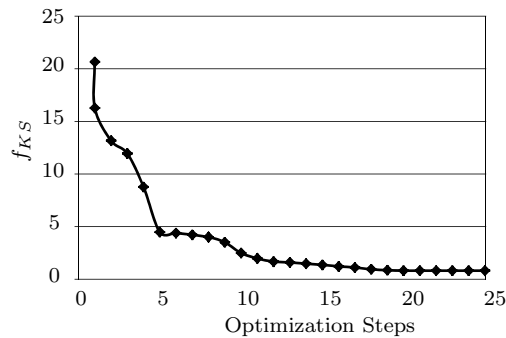

Figure 12: Optimization quadratic with two knotspans

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7. Conclusion

The benchmark examples show that the presented layout optimization technique works. The paper is intended to illustrate the possibility of using isogeometric analysis for segmentizing shell structures. There are certainly several aspects which need to be investigated in more detail e.g. multiple load cases, more complex geometries etc.. The possibility of a direct integration of the proposed optimization into the design process gives this method a promising future.

8. Acknowledgment

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