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Density filters for topology optimization based on the geometric and harmonic means

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1. Abstract

In the ground structure approach for topology optimization, some restriction method is needed to avoid mesh dependent solutions and enforce a length scale on the optimized structure. This talk presents new restriction methods in the form of four different density filters to be used within a relaxation/penalization method like SIMP. The first two are based on the geometric mean, while the last two are based on the harmonic mean. The filters have been numerically compared with other well-known filters, and the obtained results indicate that the new filters are able to generate almost black and white solutions with competitive objective function values after a competitive number of iterations.

Moreover, and perhaps more important, it is demonstrated that on some problems several of the considered filters generate different final solutions with close to equal objective values but with quite different topologies and/or geometries. In such cases, an extensive ”filter tool box” would provide the user with several different optimized solutions, each giving a (hopefully) clever suggestion of how the real structure should be designed. Since there are often some aspects which are not properly considered by the optimization model, like manufacturing aspects, it is clearly a benefit for the user to have several clever suggestions to choose between for the further processing.

It turns out that each of the four suggested new filters generates, on at least one test problem, a solution with topology and/or geometry not obtained by any of the other filters. Together with the apparent ability to produce almost black and white solutions with competitive objective function values, this should make the new filters interesting candidates for any “filter tool box” in topology optimization.

2. Keywords: Topology optimization, Density filters, Geometric mean, Harmonic mean.

3. Introduction

In topology optimization, when using a relaxation/penalization approach like SIMP, [2], filters are often applied in an attempt to avoid mesh dependence of solutions and modelling problems such as checkerboards. An often used filter is the sensitivity filter, [7], wherein a smoothing filter is applied to the derivatives of the objective function, which has proven to work well in practice. Another often used filter is the linear density filter, [3] and [4], which has good convergence properties, but may sometimes produce solutions with a relatively large number of gray element. Since this might be an unwanted property, several nonlinear density filtering techniques have been suggested to give more black and white solutions, e.g. [6], [8] and [5].

In this presentation, four nonlinear density filters based on the geometric and harmonic means are suggested and numerically compared to some existing filters. Moreover, inspired by the obtained numerical results, we suggest that any topology optimization software based on a relaxation/penaliztion approach should contain a ”filter tool box” with several different filters. The arguments are the following: When topology optimization is applied in practice for designing a new structure, the most demanding task for the user is to formulate the problem in a correct and well-defined way. This includes setting up a reasonable FE-model, and specifying relevant load cases, boundary conditions, objective function, and constraint functions. When this has been done, a topology optimization software can be applied. But then, if a filter tool box is available, it takes just a relatively small additional effort to solve the considered problem several times with alternative filters. Since the relaxed and penalized problem is intrinsically nonconvex, and since different filters restrict the set of feasible solutions in different ways, this repetition will typically provide the user with several different ”optimal solutions”, each giving a (hopefully clever) suggestion of how the structure should be designed. Since there are often some aspects which are not properly considered by the optimization model, like manufacturing aspects, it should be valuable for the user to have more than one suggestion to choose between before further post processing. This does not mean that "any" filter should be included in the tool box, but if a new filter is able to, on at least some nontrivial test problems, obtain relevant topologies not obtained by other filters, then this new filter should be a candidate for inclusion. In fact, the different filters considered in this study do indeed produce quite different solutions to the considered test problems. In particular, this holds for the suggested new filters.

4. Density variables and design variables in density filters

In standard topology optimization, there is a density variable ρ_j associated with each element j in the finite element model of the design space. The physical properties of the element, that is, the mass density and the stiffness, are then controlled by this density variable via some interpolation scheme, e.g. SIMP. When using a density filter, variables called $\tilde{\rho}_j$ are introduced, related to the variables ρ_j through a function called a filter as $\tilde{\rho} = F(\rho)$, where ρ and $\tilde{\rho}$ are vectors containing the respective variables ρ_j and $\tilde{\rho}_j$ for all elements. The quantity $\tilde{\rho}_j$, identified as the physical density of element j, is then used to interpolate material properties, while the ρ_j, now denoted design variables, are to be interpreted as non-physical variables controlling the structure indirectly. All evaluations of structural behaviour are performed using the physical densities $\tilde{\rho}_j$. The optimizer is controlling the structure indirectly via the design variables ρ_j, and all derivatives are therefore calculated through the chain rule.

5. Linear Density filter

The linear density filter, [3] and [4], is defined by:

$$\tilde{\rho}_i = \sum_j w_{ij} \rho_j \tag{1}$$

where w_{ij} are weighting factors based on the distance between element i and element j.
A reasonable requirement is that if all of the design variables are equal to a certain value, then the filtered density should also obtain this value. This motivates the condition:

$$\sum_j w_{ij} = 1, \quad \text{for all } i. \tag{2}$$

which will be assumed throughout this manuscript. The weighting factors are normally taken > 0 in some neighbourhood N_i of element i, and equal to zero outside the neighbourhood. The neighbourhood is typically circular, and its radius is commonly known as the filter radius, or simply R, in which case $N_i = \{j : d(i,j) \leq R\}$, where $d(i,j)$ is the distance between the centroids of element i and j. The most frequently used weight factors are so called conic weights, defined by:

$$w_{ij} = \frac{R - d(i,j)}{\sum_{k \in N_i} (R - d(i,k))} \quad \text{if } j \in N_i, \quad \text{while } w_{ij} = 0 \ \text{ if } j \notin N_i. \tag{3}$$

6. Sensitivity filter

In the original sensitivity filter, [7], no filtering is made of the density itself, hence $\tilde{\rho} = \rho$. However, the sensitivities (derivatives) of an objective function ϕ, are filtered using the formula

$$\widetilde{\frac{\partial \phi}{\partial \rho_i}} = \frac{1}{\rho_i} \sum_{j \in N_i} \frac{\partial \phi}{\partial \rho_j} w_{ij} \rho_j \tag{4}$$

This modification of the derivatives has proven to be effective in producing mesh-independent solutions, and it is without doubt one of the most used filters in topology optimization applications.

7. Two new filters based on the geometric mean

In the above linear density filter, (1), the density of an element is defined as a weighted arithmetic mean of the design variables in the neighborhood of the element. If the weighted arithmetic mean is replaced by the weighted geometric mean, the following filter is obtained:

$$\tilde{\rho}_i = \prod_{j \in N_i} \rho_j^{w_{ij}}. \tag{5}$$

It follows from this definition that if any design variable in the neighborhood of an element is equal to zero, the filtered density for that element will be zero. This indicates that this filter will be able to create more black and white structures than the linear density filter. The filter can be rewritten as:

$$\log(\tilde{\rho}_i) = \sum_j w_{ij} \log(\rho_j). \tag{6}$$

By switching ρ and $\tilde{\rho}$ with $1-\rho$ and $1-\tilde{\rho}$ in (6), a "reversed" geometric filter is obtained:

$$\log(1 - \tilde{\rho}_i) = \sum_j w_{ij} \log(1 - \rho_j). \tag{7}$$

This filter does the opposite, that is, if any design variable in the neighborhood of an element is equal to one, the filtered density for that element will be one. Thus, any black design variable pixel will be filtered to a black circle with radius equal to the filter radius. Since taking the logarithm of zero is not defined, we introduce a strictly positive parameter α for both filters:

$$\log(\tilde{\rho}_i + \alpha) = \sum_j w_{ij} \log(\rho_j + \alpha), \tag{8}$$

$$\log(1 - \tilde{\rho}_i + \alpha) = \sum_j w_{ij} \log(1 - \rho_j + \alpha). \tag{9}$$

For large values of α, both these geometric filters become similar to the linear density filter, but the filters are meant to be used with $\alpha < 1$. In the sequel, (8) will be called "the straight geometric filter", while (9) will be called "the reversed geometric filter".

8. Two new filters based on the harmonic mean

Arithmetic and geometric means are two of the three "Pythagorean means". The third is harmonic mean, and a filter based on the weighted harmonic mean can be formulated as:

$$\frac{1}{\tilde{\rho}_i} = \sum_j \frac{w_{ij}}{\rho_j}. \tag{10}$$

This filter also has the property that if any design variable in the neighborhood of an element is equal to zero, the filtered density for that element will be zero. By switching ρ and $\tilde{\rho}$ with $1-\rho$ and $1-\tilde{\rho}$ in (10), a "reversed" harmonic filter is obtained:

$$\frac{1}{1 - \tilde{\rho}_i} = \sum_j \frac{w_{ij}}{1 - \rho_j}, \tag{11}$$

with the property that if any design variable in the neighborhood of an element is equal to one, the filtered density for that element will be one. To avoid division by zero, the strictly positive parameter α is introduced also in these two filters:

$$\frac{1}{\tilde{\rho}_i + \alpha} = \sum_j \frac{w_{ij}}{\rho_j + \alpha}, \tag{12}$$

$$\frac{1}{1 - \tilde{\rho}_i + \alpha} = \sum_j \frac{w_{ij}}{1 - \rho_j + \alpha}. \tag{13}$$

Again, for large values of α, both these filters become similar to the linear density filter, but the filters are meant to be used with $\alpha < 1$. In the sequel, (12) will be called "the straight harmonic filter", while (13) will be called "the reversed harmonic filter".

9. Convexity and concavity properties of the new filters

The filter (1) is called *linear* since each density $\tilde{\rho}_i$ is a linear function of the variable vector ρ. The geometric and harmonic filters are *nonlinear* in this respect, but a closer examination reveals that the two reversed filters (9) and (13) are in fact *convex* density filters (each $\tilde{\rho}_i$ is a convex function of ρ), while the two straight filters (8) and (12) are *concave* density filters (each $\tilde{\rho}_i$ is a concave function of ρ). These

properties can be proved by showing (analytically) that the Hessian matrix of $\tilde{\rho}_i(\rho)$ is always positive semidefinite for the filters (9) and (13), while it is always negative semidefinite for the filters (8) and (12). Three implications of these properties are the following:
I.) For each of the convex filters, the volume $\sum_i \tilde{\rho}_i(\rho)$ is a convex function of ρ, so that the feasible region induced by a volume constraint is a convex set.
II.) For each of the concave filters, the compliance without penalization ($p = 1$ in SIMP) is a convex function of ρ.
III.) For each of the concave filters, the volume $\sum_i \tilde{\rho}_i(\rho)$ is a concave function of ρ. This implies that the MMA approximation of the volume will always be "conservative", so that the optimal solution of the MMA subproblem will always satisfy the original volume constraint.

10. Numerical test problems

The new filters have been tested on several different test problems, and their performance has been compared to several existing filters. In this presentation, however, we will present results only for three closely related test examples and only for the above six filters (4), (1), (8), (9), (12) and (13).
The SIMP approach for topology optimization is used, with element-wise constant densities.
Four-node bilinear finite elements are used, and the implementation is done in Matlab, using the "88-line code" from [1]. The optimizer used is the Matlab version of the method of moving asymptotes (MMA), [9], with default values on all parameters, except for the addition of a move limit 0.2 on each variable in each iteration, implemented through the parameters `alfa` and `beta` in the subroutine `mmasub.m`. (The same move limit as used in the OC method of the "88-line code" in [1].)
In each of the three presented test problems, the sum of the compliances corresponding to four different load cases is minimized subject to a volume constraint. The only difference between the test problems is the weightings of the four load cases.
The design domain consists of $140 \times 140 = 19600$ square elements and $141 \times 141 = 19881$ nodes with coordinates (x, y), where $x \in \{-70, -69, \ldots, 69, 70\}$ and $y \in \{-70, -69, \ldots, 69, 70\}$.
There are 18 degrees of freedom which are fixed to zero, namely the degrees of freedom corresponding to the nine nodes with coordinates (x, y), where $x \in \{-1, 0, 1\}$ and $y \in \{-1, 0, 1\}$.
In each of the four load cases, there are applied external forces at each of the twelve nodes with coordinates (x_i, y_i) according to Table 1. Note that $(30, 52) \approx 60 \cdot (1/2, \sqrt{3}/2)$, which means that the twelve nodes are located approximately on the boundary of a circle with radius 60 (with one "hour" between each node).
The force $(f^x_{\ell i}, f^y_{\ell i})$ applied at (x_i, y_i) for the different load cases is defined in Table 2 and illustrated in Figure 1, and the load coefficients c_ℓ in Table 2 are chosen according to Table 3.
The optimization problem can be formulated as:

$$\text{minimize} \quad \sum_{\ell=1}^{4} f_\ell^{\mathsf{T}} u_\ell(\rho) \quad \text{subject to} \quad \sum_{i=1}^{n} \tilde{\rho}_i(\rho) \leq V^*, \quad \rho \in [0, 1]^n, \tag{14}$$

where, for a given $\rho \in [0, 1]^n$, the displacement vector $u_\ell(\rho)$ is obtained as the solution to $K(\rho)u_\ell = f_\ell$, where $K(\rho) = \sum_{i=1}^{n}(E_{\min} + \tilde{\rho}_i(\rho)^p(E_0 - E_{\min}))K_i$. Here, u_ℓ and f_ℓ are the displacement and force vectors corresponding to load case ℓ, K_i is the stiffness matrix of element i, $E_{\min} = 10^{-9}$ and $E_0 = 1$. The physical densities $\tilde{\rho}_i(\rho)$ are obtained from one of the equations (1), (8), (9), (12), (13), or, if using the Sensitivity filter, $\tilde{\rho}_i(\rho) = \rho_i$. The SIMP penalty parameter is $p = 4$. All the six considered filters use conic weights with filter radius 2.4 elements (= 4% of the radius of the "circle of loads"). In the geometric and harmonic filters, the parameter α has been chosen such that if $\sum_j w_{ij}\rho_j = 0.5$ then $\tilde{\rho}_i$ becomes roughly 0.1 for the straight filters and 0.9 for the reversed filters. More precisely, $\alpha = 0.1$ for the two harmonic filters, and $\alpha = 0.01$ for the two geometric filters.
Concerning the convergence criterion, the iterations are stopped when no variable has changed by more than 0.001 since the previous iteration, which is somewhat harder than the more commonly used 0.01.
Since the objective value of an obtained solution is very sensitive to the amount of (heavily penalized) gray structure, the linear density filter almost always comes out as a loser when comparing objective value, even though the generated topology and geometry may in fact be very good. Since it may be more interesting to compare the quality of different topologies and geometries rather than the amount of gray, the obtained final solutions for the different filters have also been rounded to pure black and white solutions as follows: The right hand side of the volume constraint in the SIMP problem is set to 0.333 times the total number of elements, i.e., $0.333 \cdot 19600 = 6526.8$. When the convergence criterion has been fulfilled and the iterations stopped, the obtained (slightly gray) solution is rounded to a completely black and white solution by letting the 6528 elements with largest values on their physical design variable

(”density”) be black, and the other 13072 elements white. This means that $6528/19600 \approx 0.33306$ of the total design domain becomes black. The obtained topologies turns out to be essentially unaltered by this final rounding, but the boundaries become much sharper (of course) and the compliances decrease.
Finally, it should noted that symmetry of the structure is *not* enforced.
The obtained results on the threes test problems are presented in Tables 4–6 and Figures 2–4 where the physical densities $\tilde{\rho}_i$ are plotted. Each table contains the following information:
Column 1: Number of MMA iterations for obtaining the gray solution.
Column 2: Penalized objective value for the obtained gray structure divided by objective value for the completely black structure (19600 black elements).
Column 3: Objective value for the rounded solution (6528 black elements) divided by objective value for the completely black structure (19600 black elements).
Column 4: Number of holes in the rounded solution.
A comment is perhaps in order regarding the result of the sensitivity filter on Test problem 2. While providing excellent solutions to Test problems 1 and 3, the sensitivity filter converges to an almost completely gray solution to Test problem 2. Actually, this solution appears to be a KKT point with respect to the filtered derivatives, but it is not a local optimum to the considered problem. In fact, looking more closely at the iteration history, there are, among the earlier iterates, feasible solutions with lower objective values than the obtained final objective value. It is unclear why Test problem 2 causes this behaviour, but it shows that even established filters may, on certain problems, encounter difficulties not encountered by other filters. Again, this is an argument for a filter tool box.

11. Conclusions

One must be extremely careful to draw any conclusions from such a limited number of numerical tests, but two things can be claimed with certainty: 1. There are topology optimization problems to which the different filters produce quite different topologies, with different number of holes and quite different objective values of the gray solutions, but with objective values of the corresponding rounded solutions which are very close to each other. 2. There are topology optimization problems for which relatively small changes in the load conditions completely change both the obtained topologies and the relative ranking of the compared filters. Both these conclusions support the suggestion of a “filter tool box”.
As a final (non-scientific) comment, we think that the obtained results indicate that the suggested new filters should be included in such a tool box.

12. References

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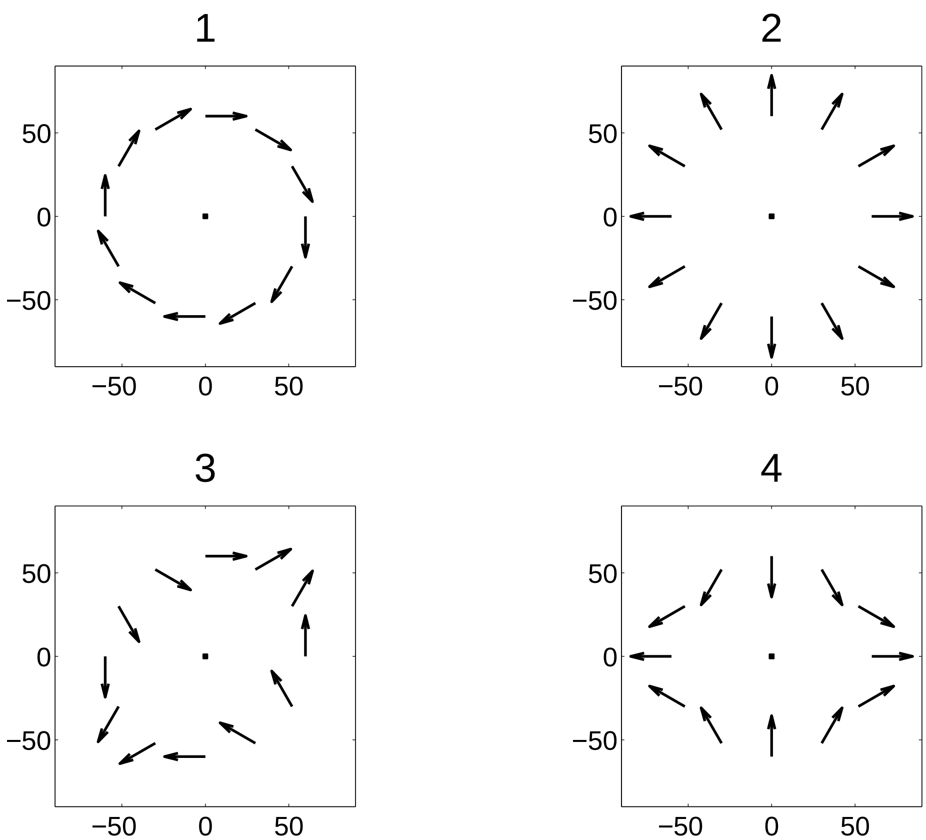


Figure 1: Directions of the applied external forces in the four load cases

Table 1: Coordinates for nodes where the external forces are applied

i	1	2	3	4	5	6	7	8	9	10	11	12
x_i	30	52	60	52	30	0	-30	-52	-60	-52	-30	0
y_i	52	30	0	-30	-52	-60	-52	-30	0	30	52	60

Table 2: External forces at the node with coordinates (x_i, y_i)

Load case:	1	2	3	4
$f_{\ell i}^x$	$c_1 y_i$	$c_2 x_i$	$c_3 y_i$	$c_4 x_i$
$f_{\ell i}^y$	$-c_1 x_i$	$c_2 y_i$	$c_3 x_i$	$-c_4 y_i$

Table 3: Load coefficients c_ℓ in the previous table

	c_1	c_2	c_3	c_4
Test problem 1:	0.0001	0.01	0.01	0.01
Test problem 2:	0.0001	0.02	0.01	0.01
Test problem 3:	0.0001	0.02	0.02	0.01

Table 4: Results for Test problem 1

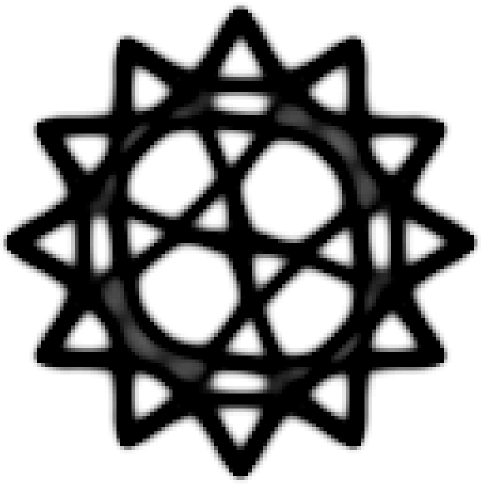
Filter	Iter	Gray	Rounded	Holes
Sensitivity filter	616	3.23	2.68	26
Linear Density	1112	3.49	2.68	25
Straight Geometric	396	3.23	2.68	26
Reversed Geometric	422	3.35	2.67	36
Straight Harmonic	209	3.30	2.68	32
Reversed Harmonic	842	3.55	2.73	38

Table 5: Results for Test problem 2

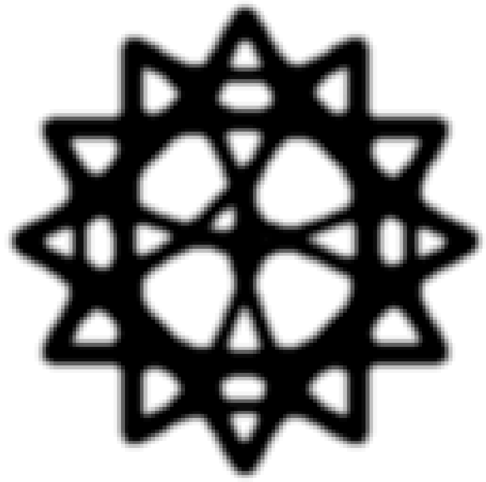
Filter	Iter	Gray	Rounded	Holes
Sensitivity filter	176	5.70	$> 10^4$	1
Linear Density	379	3.31	2.61	17
Straight Geometric	430	3.13	2.60	28
Reversed Geometric	260	3.12	2.60	24
Straight Harmonic	341	3.17	2.62	28
Reversed Harmonic	1018	3.13	2.58	16

Table 6: Results for Test problem 3

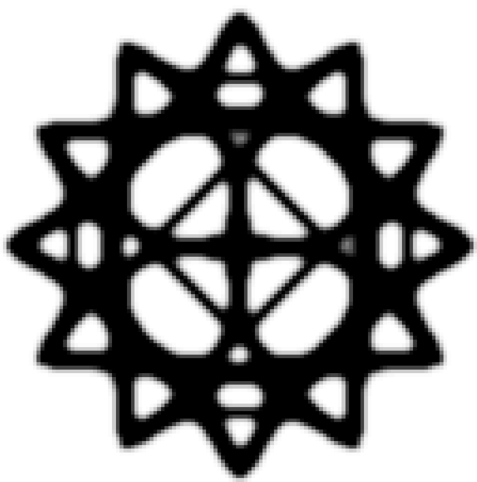
Filter	Iter	Gray	Rounded	Holes
Sensitivity filter	753	3.11	2.58	36
Linear Density	199	3.47	2.59	32
Straight Geometric	202	3.18	2.60	36
Reversed Geometric	335	3.13	2.63	16
Straight Harmonic	173	3.21	2.62	36
Reversed Harmonic	407	3.28	2.61	24



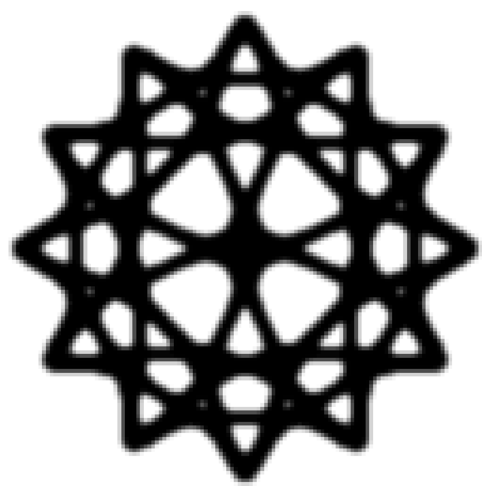
a) Sensitivity filter



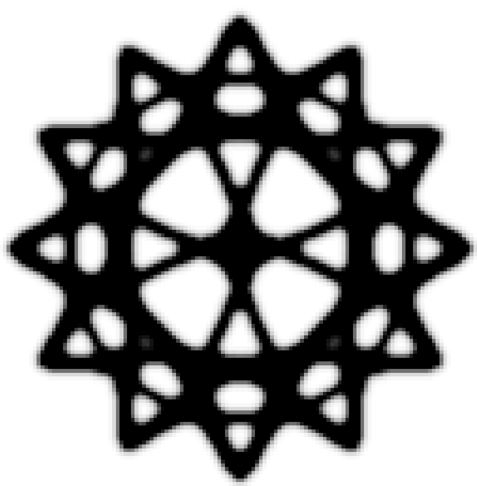
b) Linear Density filter



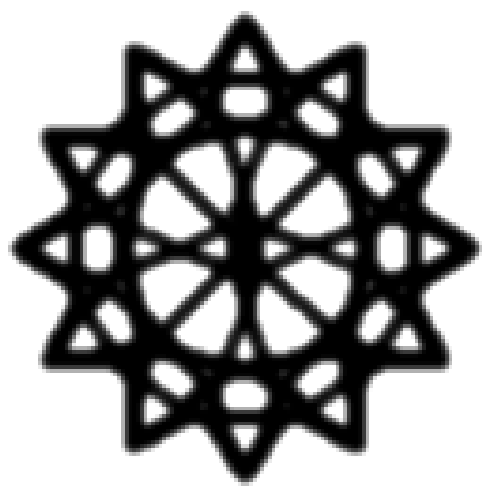
c) Straight Geometric filter



d) Reversed Geometric filter

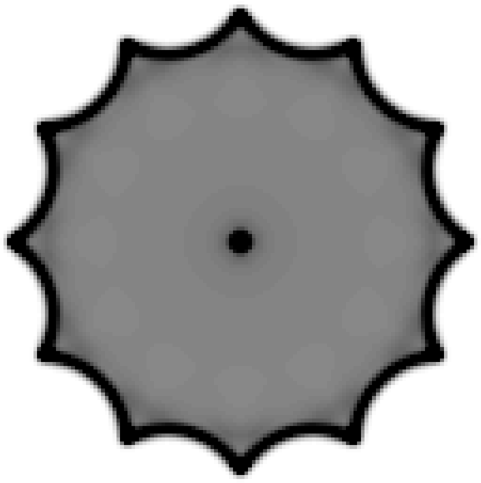


e) Straight Harmonic filter

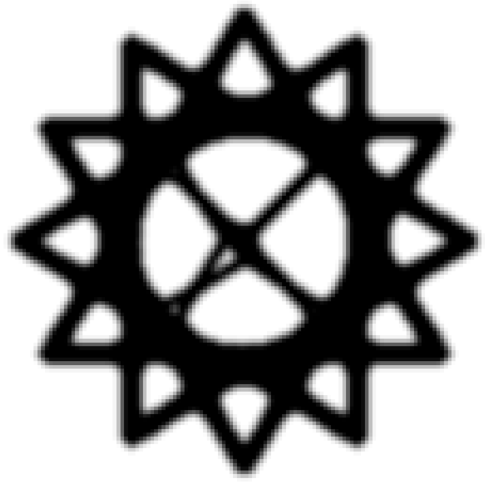


f) Reversed Harmonic filter

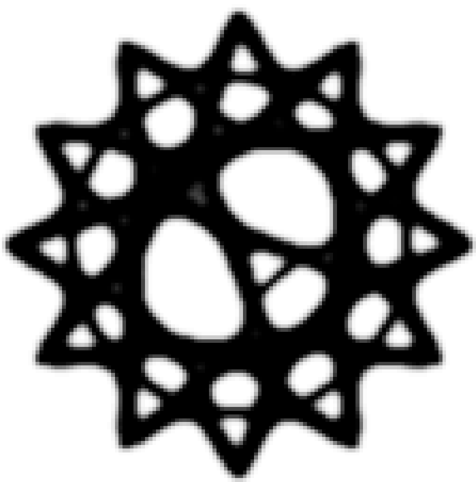
Figure 2: Obtained density distributions for Test problem 1



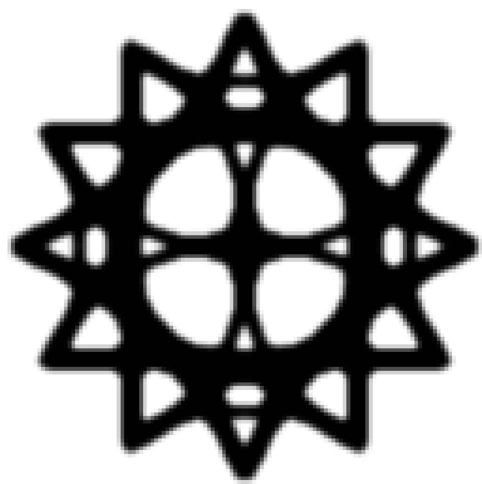
a) Sensitivity filter



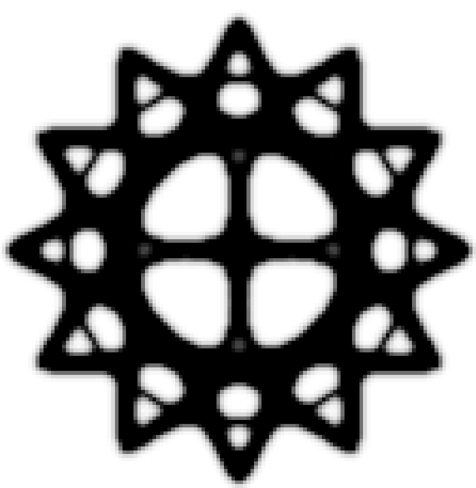
b) Linear Density filter



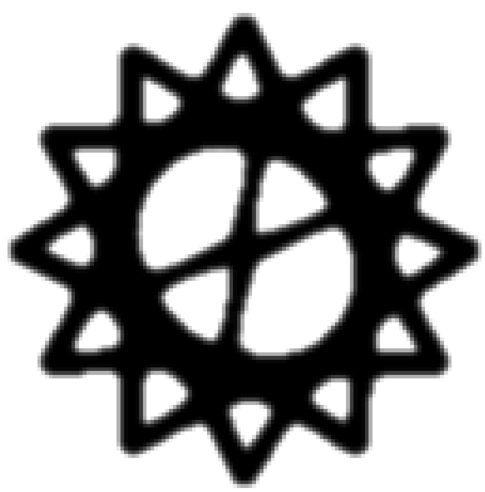
c) Straight Geometric filter



d) Reversed Geometric filter

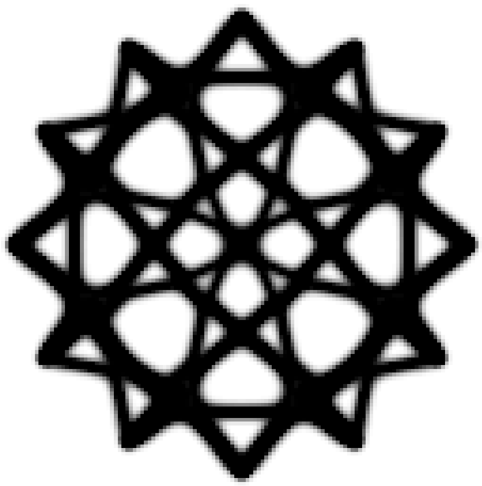


e) Straight Harmonic filter

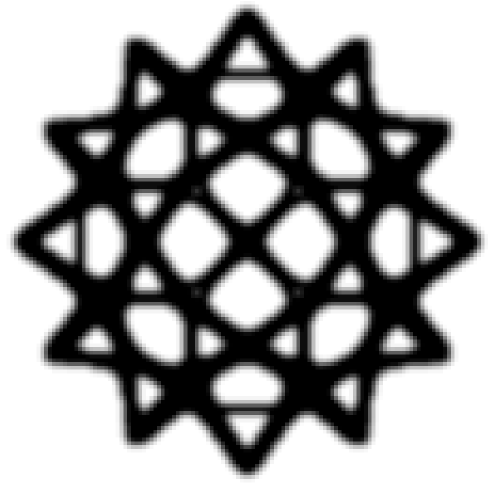


f) Reversed Harmonic filter

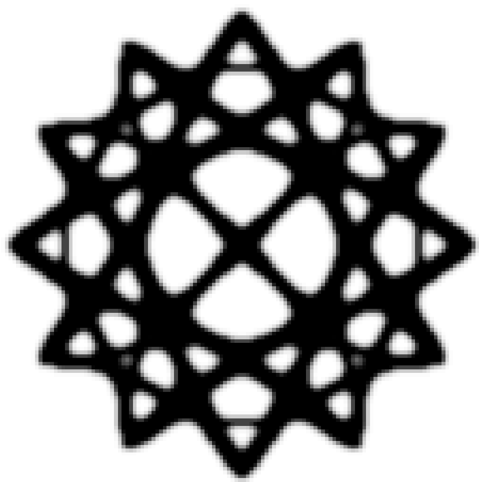
Figure 3: Obtained density distributions for Test problem 2



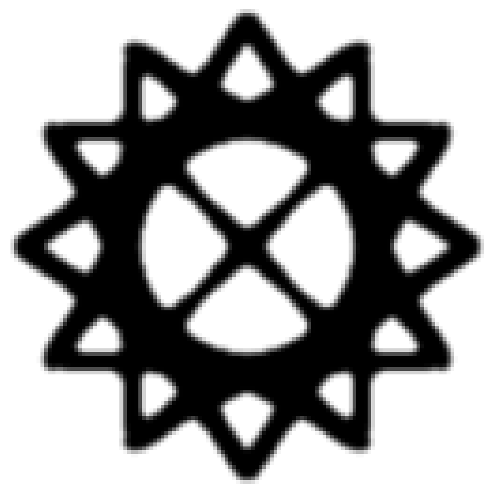
a) Sensitivity filter



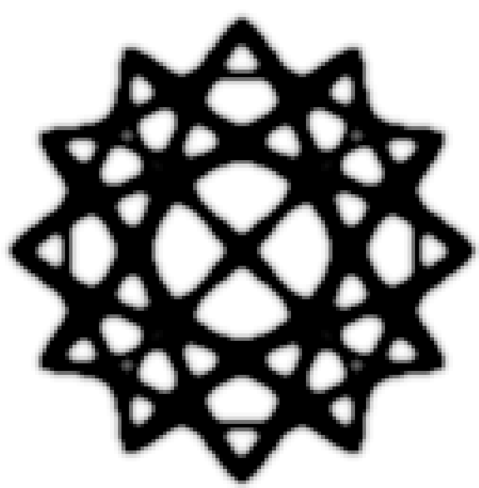
b) Linear Density filter



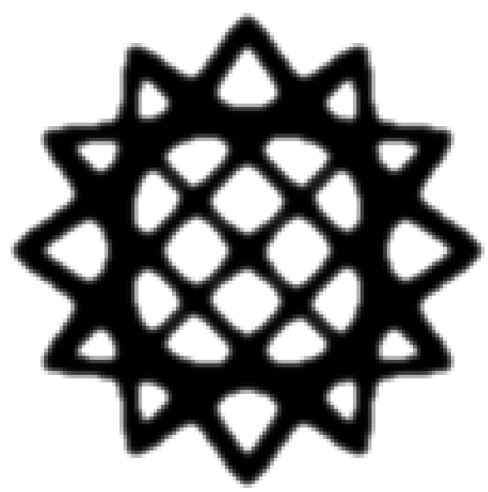
c) Straight Geometric filter



d) Reversed Geometric filter



e) Straight Harmonic filter



f) Reversed Harmonic filter

Figure 4: Obtained density distributions for Test problem 3