**10th World Congress on Structural and Multidisciplinary Optimization**
May 19 -24, 2013, Orlando, Florida, USA

# Sampling-based Approach for Design Optimization in the Presence of Interval Variables

**David Yoo[1] and Ikjin Lee[2]**

[1] University of Connecticut, Storrs, Connecticut, USA, david.yoo@engr.uconn.edu
[2] University of Connecticut, Storrs, Connecticut, USA, ilee@engr.uconn.edu

## 1. Abstract
This paper proposes a methodology for sampling-based design optimization in the presence of interval variables. Assuming that an accurate surrogate model is available, the proposed method first searches the worst combination of interval variables for constraints when only interval variables are present or for probabilistic constraints when both interval and random variables are present. Due to the fact that the worst combination of interval variables for probability of failure does not always coincide with that for a performance function, the proposed method directly uses the probability of failure to obtain the worst combination of interval variables when both interval and random variables are present. To calculate sensitivities of the constraints and probabilistic constraints with respect to interval variables by the sampling-based method, behavior of interval variables at the worst case is defined by the Dirac delta function. Then, Monte Carlo simulation is applied to calculate the constraints and probabilistic constraints with the worst combination of interval variables, and their sensitivities. The important merit of the proposed method is that it does not require gradients of performance functions and transformation from X-space to U-space for reliability analysis after the worst combination of interval variables is obtained, thus there is no approximation or restriction in calculating sensitivities of constraints or probabilistic constraints. Numerical results indicate that the proposed method can search the worst case probability of failure with both efficiency and accuracy and that it can perform design optimization with mixture of random and interval variables by utilizing the worst case probability of failure search.
**2. Keywords:** Interval Variables, Sampling-Based Method, Dirac Delta function, Monte Carlo simulation, Surrogate model

## 3. Introduction
Reliability analysis and reliability-based design optimization (RBDO) have been developed to take uncertainty into consideration, and have been successfully adapted to many engineering applications such as crashworthiness of vehicle and structural-acoustic system design [1-10]. The uncertainty is generally categorized into aleatory and epistemic uncertainties, where aleatory uncertainty is considered as irreducible whereas epistemic uncertainty is reducible by collecting more data. In case when sufficient amount of data for statistical information is unavailable, possibility-based (or fuzzy set) methods have utilized membership function to model insufficiently collected data [11], and adjusted standard deviation and correlation coefficient involving confidence intervals have been utilized to offset an inaccurate modeling of data [12,13]. When degree of insufficiency of data is even greater as only lower and upper bounds of data are available, the methods listed above are not applicable anymore, thus the different approach is required.
To deal with data of which only lower and upper bounds are available, a method of multi-point approximation that evaluates the weighting function and local approximations separately has been first developed for interval analysis [14]. Then, the most probable point (MPP) based first-order reliability method (FORM) has been utilized for design optimization with mixture of random and interval variables [15]. As bounds of probability of failure or reliability exist in the presence of interval variables, design optimization for the worst and best cases has been also developed [16], and sensitivity analysis considering bounds of interval variables and probability of failure has been developed accordingly [17].
By using the MPP-based FORM, a design optimum is very efficiently searched; however it is generally less accurate for highly nonlinear performance functions and high-dimensional input variables [18-21]. To improve the accuracy on this occasion, the second order reliability method (SORM) can be applied after the MPP search; however, its efficiency is sacrificed due to the fact that computation of the Hessian matrix is required by the SORM [22-25]. The MPP-based dimension reduction method (DRM) can be also used for approximately assessing the reliability of a system, which is used as a probabilistic constraint in RBDO [26-28].
In absence of accurate sensitivities of performance functions, the MPP-based reliability analysis or RBDO, which utilizes sensitivities of performance functions to find the MPP, cannot be directly used, instead the sampling-based reliability analysis or RBDO can be used [29-31]. Assuming an accurate surrogate model is given [32-36], Monte Carlo simulation (MCS) [37] can be applied to find a design optimum with affordable computational burden.
This study introduces interval analysis and design optimization utilizing the sampling-based method in the

presence of only interval variables and in the presence of both random and interval variables. Due to the presence of interval variables, obtaining the worst combination of interval variables for both constraints and probabilistic constraints is involved [15]. When both random and interval variables are present, the worst combination of interval variables for probability of failure is directly searched using the probability of failure and its sensitivity since the design point where the worst case probability of failure occurs does not always coincide with that for the worst case performance function; it is highly likely as many studies have assumed, however not always. To evaluate sensitivities of probability of failure with respect to interval variables, the Dirac delta function is utilized to define behavior of the interval variables at the worst case [38-41].

Assuming an accurate surrogate model is given, one merit of the proposed method exists not only during the worst case probability of failure search but also during reliability analysis after the worst case probability of failure search. The worst case probability of failure search, which will be explained in Section 4.2, utilizes a vector of interval variables instead of individual components of the vector, and it thus promises efficiency. Also, it resolves the problem that the worst case probability of failure does not always occur where the worst case performance occurs. During the reliability analysis after the worst case probability of failure search, another merit of the proposed method is that it does not make further approximations since it does not require gradients of the performance function and transformation of design variables from X-space to U-space, thus there is no approximation or restriction in calculating the sensitivities of constraints or probabilistic constraints [30].

The paper is organized as follow. Section 4 briefly reviews the sampling-based RBDO. Section 5 explains design optimization with interval variables only, including the algorithm to obtain the worst combination of interval variables for a performance function, mathematical derivation for sensitivities of each constraint with respect to interval variables by defining behavior of the interval variables at the worst case using the Dirac delta function, and their computation by the MCS. Section 6 explains the sampling-based design optimization with both random and interval variables including details to obtain the worst combination of interval variables for probability of failure and computation of probabilistic constraints and their sensitivities by the MCS. Section 7 illustrates search for the worst combination of interval variables for probability of failure and design optimization with random and interval variables using numerical examples. Section 8 summarizes and concludes the paper with discussion of future research.

## 4. Review of Sampling-based RBDO

### 4.1 Formulation of RBDO

The mathematical formulation of RBDO is expressed as

$$\begin{aligned} &\text{minimize} && \text{cost}(\mathbf{d}) \\ &\text{subject to} && P\left[G_j\left(\mathbf{X^R}\right)>0\right]\le P_{F_j}^{\text{tar}}, \quad j=1,...,\text{NC} \\ & && \mathbf{d}^{\text{L}}\le\mathbf{d}\le\mathbf{d}^{\text{U}},\ \ \mathbf{d}\in\mathbf{R}^{\text{ndv}},\ \text{and}\ \mathbf{X^R}\in\text{R}^{NR} \end{aligned} \tag{1}$$

where $\mathbf{d}=\boldsymbol{\mu}(\mathbf{X})$ is the design vector, which is the mean value of the $NR$-dimensional random vector $\mathbf{X^R}=\left\{X_1^{\text{R}},X_2^{\text{R}},...X_{NR}^{\text{R}}\right\}^{\text{T}}$; $P_{F_j}^{\text{tar}}$ is the target probability of failure for the $j^{\text{th}}$ constraint; and NC, ndv, and $NR$ are the number of probabilistic constraints, design variables, and random variables, respectively [30].

To carry out RBDO using Eq. (1), the probabilistic constraints and their sensitivities must be evaluated. Reviews on the reliability and its sensitivity analyses are explained in Sections 2.2 and 2.3, respectively.

### 4.2 Probability of Failure

The probability of failure with random variables, denoted by $P_F$, is defined using a multi-dimensional integral

$$P_F\left(\boldsymbol{\psi}\right)\equiv P\left[\mathbf{X^R}\in\Omega_F\right]=\int_{\text{R}^{NR}} I_{\Omega_F}\left(\mathbf{x^R}\right)f_{\mathbf{X^R}}\left(\mathbf{x^R};\boldsymbol{\psi}\right)d\mathbf{x^R}=E\left[I_{\Omega_F}\left(\mathbf{X^R}\right)\right] \tag{2}$$

where $\boldsymbol{\psi}$ is a matrix of distribution parameters, which includes mean ($\boldsymbol{\mu}$) and/or standard deviation ($\boldsymbol{\sigma}$) of $\mathbf{X^R}$; $P[\bullet]$ represents a probability measure; $\Omega_F$ is defined as a failure set; $f_{\mathbf{X^R}}\left(\mathbf{x^R};\boldsymbol{\psi}\right)$ is a joint probability density function (PDF) of $\mathbf{X^R}$; and $E[\bullet]$ represents the expectation operator [30,42]. $I_{\Omega_F}\left(\mathbf{x^R}\right)$ in Eq. (2) is called an indicator function and defined as

$$I_{\Omega_F}\left(\mathbf{X}^{\mathbf{R}}\right) \equiv \begin{cases} 1, & \mathbf{x}^{\mathbf{R}} \in \Omega_F \\ 0, & \text{otherwise} \end{cases}. \tag{3}$$

**4.3 Sensitivity of Probability of Failure**

With the four regularity conditions satisfied, which are also explained in detail in Ref. [30], taking the partial derivative of Eq. (2) with respect to $\mu_i$ yields

$$\frac{\partial P_F(\boldsymbol{\mu})}{\partial \mu_i} = \frac{\partial}{\partial \mu_i} \int_{\mathbb{R}^{NR}} I_{\Omega_F}\left(\mathbf{x}^{\mathbf{R}}\right) f_{\mathbf{X}^{\mathbf{R}}}\left(\mathbf{x}^{\mathbf{R}};\boldsymbol{\mu}\right) d\mathbf{x}^{\mathbf{R}} \tag{4}$$

and the differential and integral operators can be interchanged due to the 4th regularity condition in Ref. [30] and the Lebesgue dominated convergence theorem [43,44] giving

$$\frac{\partial P_F(\boldsymbol{\mu})}{\partial \mu_i} = E\left[ I_{\Omega_F}\left(\mathbf{X}^{\mathbf{R}}\right) \frac{\partial \ln f_{\mathbf{X}^{\mathbf{R}}}\left(\mathbf{X}^{\mathbf{R}};\boldsymbol{\mu}\right)}{\partial \mu_i} \right]. \tag{5}$$

The partial derivative of the log function of the joint PDF in Eq. (5) with respect to $\mu_i$ is known as the first-order score function [30] for $\mu_i$ and is denoted as

$$s_{\mu_i}^{(1)}\left(\mathbf{X}^{\mathbf{R}};\boldsymbol{\mu}\right) \equiv \frac{\partial \ln f_{\mathbf{X}^{\mathbf{R}}}\left(\mathbf{X}^{\mathbf{R}};\boldsymbol{\mu}\right)}{\partial \mu_i}. \tag{6}$$

To derive the sensitivity of the probability of failure in Eq. (2), it is required to know the first-order score function in Eq. (6), which is obtained using the following equation for independent random variables

$$s_{\mu_i}^{(1)}\left(\mathbf{X}^{\mathbf{R}};\boldsymbol{\mu}\right) \equiv \frac{\partial \ln f_{\mathbf{X}^{\mathbf{R}}}\left(\mathbf{X}^{\mathbf{R}};\boldsymbol{\mu}\right)}{\partial \mu_i} = \frac{\partial \ln f_{X_i^{\mathrm{R}}}\left(x_i^{\mathrm{R}};\mu_i\right)}{\partial \mu_i} \tag{7}$$

where $f_{X_i^{\mathrm{R}}}\left(x_i^{\mathrm{R}};\mu_i\right)$ is the marginal PDF corresponding to the $i$th random variable $X_i^{\mathrm{R}}$, and obtained using the following equation for correlated random variables

$$s_{\mu_i}^{(1)}\left(\mathbf{X}^{\mathbf{R}};\boldsymbol{\mu}\right) \equiv \frac{\partial \ln f_{\mathbf{X}^{\mathbf{R}}}\left(\mathbf{X}^{\mathbf{R}};\boldsymbol{\mu}\right)}{\partial \mu_i} = \frac{\partial \ln c(u,v;\theta)}{\partial \mu_i} + \frac{\partial \ln f_{X_i^{\mathrm{R}}}\left(x_i^{\mathrm{R}};\mu_i\right)}{\partial \mu_i} \tag{8}$$

where $c$ is a copula density function, $u = F_{X_i^{\mathrm{R}}}\left(x_i^{\mathrm{R}};\mu_i\right)$ and $v = F_{X_j^{\mathrm{R}}}\left(x_j^{\mathrm{R}};\mu_j\right)$ are marginal CDFs for $X_i^{\mathrm{R}}$ and $X_j^{\mathrm{R}}$, respectively, and $\theta$ is the correlation coefficient between $X_i^{\mathrm{R}}$ and $X_j^{\mathrm{R}}$ [30]. The information of marginal PDFs, CDFs, and commonly used copula density functions is listed in detail in Ref. [30].

**4.4 Simplification of the nonlinear characteristics of vehicle behavior**

The MCS can be applied to calculate the probabilistic constraints in Eq. (1) and their sensitivities. Denoting a surrogate model for the $j$th constraint function with random variables as $\hat{G}_j\left(\mathbf{X}^{\mathbf{R}}\right)$, the probabilistic constraints in Eq. (1) can be calculated as

$$P_{F_j} \equiv P\left[G_j\left(\mathbf{X}^{\mathbf{R}}\right) > 0\right] \cong \frac{1}{K}\sum_{k=1}^{K} I_{\hat{\Omega}_{F_j}}\left[\mathbf{X}^{\mathbf{R}(k)}\right] \le P_{F_j}^{\mathrm{tar}} \tag{9}$$

where $K$ is the MCS sample size, $\mathbf{X}^{\mathbf{R}(k)}$ is the $k$th realization of $\mathbf{X}^{\mathbf{R}}$, and the failure set $\hat{\Omega}_{F_j}$ for the surrogate model is defined as $\hat{\Omega}_{F_j} \equiv \left[ \mathbf{X}^{\mathbf{R}} : \hat{G}_j\left(\mathbf{x}^{\mathbf{R}}\right) > 0 \right]$ [30]. Sensitivities of the probabilistic constraints in Eq. (1) are calculated using the score function as

$$\frac{\partial P_{F_j}}{\partial \mu_i} \cong \frac{1}{K} \sum_{k=1}^{K} I_{\hat{\Omega}_{F_j}} \left[ \mathbf{X}^{\mathbf{R}(k)} \right] s_{\mu_i}^{(1)} \left[ \mathbf{X}^{\mathbf{R}(k)}; \boldsymbol{\mu} \right] \tag{10}$$

where $s_{\mu_i}^{(1)} \left[ \mathbf{X}^{\mathbf{R}(k)}; \boldsymbol{\mu} \right]$ is obtained using Eqs. (7) and (8) for independent and correlated random variables, respectively.

## 5. Design Optimization with Interval Variables

### 5.1 Formulation of Design Optimization with Interval Variables

The mathematical formulation of design optimization with interval variables only is expressed as

$$\begin{aligned} &\text{minimize} && \text{cost}(\mathbf{d}) \\ &\text{subject to} && G_j\left(\mathbf{X}_j^{\mathbf{I,worst}}\right) < 0, \quad j = 1, \ldots, \text{NC} \\ & && \mathbf{d}^{\text{L}} \le \mathbf{d} \le \mathbf{d}^{\text{U}}, \quad \mathbf{d} \in \text{R}^{\text{ndv}}, \quad \text{and} \quad \mathbf{X}^{\mathbf{I}} \in \text{R}^{NI} \end{aligned} \tag{11}$$

where $\mathbf{d} = \overline{\mathbf{X}^{\mathbf{I}}}$ is the design vector, which is the mid-point of the *NI*-dimensional interval vector $\mathbf{X}^{\mathbf{I}} = \left\{ X_1^{\text{I}}, X_2^{\text{I}}, \ldots, X_{NI}^{\text{I}} \right\}^{\text{T}}$ where *NI* is the number of interval variables. $\mathbf{X}_j^{\mathbf{I,worst}}$ in Eq. (11) is the worst case interval variables for the $j^{\text{th}}$ constraint, which is obtained by solving the optimization problem to

$$\begin{aligned} &\text{maximize} && G_j\left(\mathbf{X}^{\mathbf{I}}\right) \\ &\text{subject to} && \left| X_i^{\text{I}} - \overline{X_i^{\text{I}}} \right| \le \frac{\delta_i^{\text{I}}}{2} \;\; \textit{for} \;\; i = 1, \ldots, NI \end{aligned} \tag{12}$$

where $\delta_i^{\text{I}}$ is the interval length of $X_i^{\text{I}}$. It should be noted that as any statistical information of an interval variable $\mathbf{X}^{\mathbf{I}}$ is not available, $\mathbf{X}^{\mathbf{I,worst}}$ must be considered for the design optimization.

To carry out the design optimization with interval variables using Eq. (11), constraints with the worst case interval variables, namely the worst case constraints or the worst case performance, and their sensitivities must be evaluated. Each of the worst case constraint is obtained by the worst case performance search that solves Eq. (12) and will be explained in Section 3.2, and sensitivity analysis of each of the worst case constraint and its calculation are explained in Section 3.3. It is assumed in this study that gradients of performance functions are not available; however it can be directly used if available.

### 5.2 Algorithm Searching for Worst Case Performance

The algorithm explained in this section searches the worst case performance, and the algorithm was originally developed by Liu *et al*. in Ref. 11 for the maximal possibility search (MPS) for possibility-based design optimization. An important merit of the proposed algorithm is that it utilizes a vector of interval variables and a vector of sensitivities of a performance function with respect to all interval variables, thus its efficiency is not affected by the dimension of the interval variables. The algorithm for the worst case performance search is summarized as following, which is also shown in the flowchart in Fig. 1.

*Step 1.* Normalize interval variables $X_i^{\text{I}}$ using

$$Z_i^{\text{I}} = \frac{X_i^{\text{I}} - \overline{X_i^{\text{I}}}}{\delta_i^{\text{I}}} \;\; \text{or} \;\; X_i^{\text{I}} = \overline{X_i^{\text{I}}} + \delta_i^{\text{I}} \cdot Z_i^{\text{I}} \tag{13}$$

such that $\left| Z_i^{\text{I}} \right| \le 0.5$.

*Step 2.* Set the iteration counter $k$ = 0 with the convergence parameter $\varepsilon$ = $10^{-3}$. Set $j$ = 1. Let $\mathbf{Z}^{\mathbf{I}(0)} = \mathbf{0}$. Calculate the performance $G\left(\mathbf{Z}^{\mathbf{I}(0)}\right)$ and the sensitivity $\nabla G\left(\mathbf{Z}^{\mathbf{I}(0)}\right)$. It is explained in Section 3.3 how to obtain $\nabla G\left(\mathbf{Z}^{\mathbf{I}(0)}\right)$. Let the direction vector be $\mathbf{d}^{(0)} = \nabla G\left(\mathbf{Z}^{\mathbf{I}(0)}\right)$.

*Step 3.* Search the next point as $\mathbf{Z}^{\mathbf{I}(k+1)} = 0.5 \cdot \operatorname{sgn}\left(\mathbf{d}^{(k)}\right)$ where 0.5 is obtained from *Step 1*. Let $k = k + 1$.

*Step 4.* Calculate the performance $G\left(\mathbf{Z}^{\mathbf{I}(k)}\right)$ and its sensitivity $\nabla G\left(\mathbf{Z}^{\mathbf{I}(k)}\right)$. Let a conjugate direction vector $\mathbf{d}^{(k)} = \nabla G\left(\mathbf{Z}^{\mathbf{I}(k)}\right) + \beta \mathbf{d}^{(k-1)}$ where $\beta = \left(\left\|\nabla G\left(\mathbf{Z}^{\mathbf{I}(k)}\right)\right\| / \left\|\nabla G\left(\mathbf{Z}^{\mathbf{I}(k-1)}\right)\right\|\right)^2$. If $\operatorname{sgn}\left(\nabla G\left(\mathbf{Z}^{\mathbf{I}(k)}\right)\right) = \operatorname{sgn}\left(\mathbf{Z}^{\mathbf{I}(k)}\right)$, it is the worst case and go to *Step 11*.

*Step 5.* If $G\left(\mathbf{Z}^{\mathbf{I}(k)}\right) \geq G\left(\mathbf{Z}^{\mathbf{I}(j)}\right)$, let $j = k$ and go to *Step 3*. Otherwise, go to *Step 6* with $\mathbf{Z}^{\mathbf{I}(j)}$, $G\left(\mathbf{Z}^{\mathbf{I}(j)}\right)$ and $\nabla G\left(\mathbf{Z}^{\mathbf{I}(j)}\right)$.

If behavior of the performance function is not monotonic within an interval domain, in other words, if any component of the worst case interval vector does not occur at the vertex of its interval domain, interpolation algorithm must be additionally applied to obtain more accurate worst case performance [11].

*Step 6.* Let $l$ = 0 and a direction vector be $\mathbf{d}^{(l)} = \nabla G\left(\mathbf{Z}^{\mathbf{I}(j)}\right)$.

*Step 7.* Calculate the new point $\mathbf{Z}^{\mathbf{I}(k+1)}$ on the boundary of the domain from the start point $\mathbf{Z}^{\mathbf{I}(j)}$ along the search direction $\mathbf{d}^{(l)}$. Let $k = k + 1$.

*Step 8.* Calculate the performance $G\left(\mathbf{Z}^{\mathbf{I}(k)}\right)$ and its sensitivity $\nabla G\left(\mathbf{Z}^{\mathbf{I}(k)}\right)$. If

$$\begin{cases} \operatorname{sgn}\left(\partial G\left(\mathbf{Z}^{\mathbf{I}(k)}\right) / \partial\left(Z_i^{\mathrm{I}}\right)\right) = \operatorname{sgn}\left(Z_i^{\mathrm{I}}\right), & \text{for } \left|Z_i^{\mathrm{I}}\right| = 0.5 \\ \left|\partial G\left(\mathbf{Z}^{\mathbf{I}(k)}\right) / \partial\left(Z_i^{\mathrm{I}}\right)\right| < \varepsilon, & \text{for } \left|Z_i^{\mathrm{I}}\right| < 0.5 \end{cases}$$

then it is the worst case and go to *Step 11*. Otherwise, go to *Step 9*.

*Step 9.* Use $G\left(\mathbf{Z}^{\mathbf{I}(j)}\right)$, $G\left(\mathbf{Z}^{\mathbf{I}(k)}\right)$, $\nabla G\left(\mathbf{Z}^{\mathbf{I}(j)}\right)$, and $\nabla G\left(\mathbf{Z}^{\mathbf{I}(k)}\right)$ to construct the third order polynomial $f(t)$ on the straight line between $\mathbf{Z}^{\mathbf{I}(j)}$ and $\mathbf{Z}^{\mathbf{I}(k)}$ where $t$ is the parameter for the line. Calculate the maximum point $t^*$ for this polynomial. Let $\mathbf{Z}^{\mathbf{I}(k+1)}$ be the point on the line corresponding to $t^*$. Let $k = k + 1$.

*Step 10.* Calculate the performance $G\left(\mathbf{Z}^{\mathbf{I}(k)}\right)$ and its sensitivity $\nabla G\left(\mathbf{Z}^{\mathbf{I}(k)}\right)$. Check the convergence criteria using the equation in *Step 8*. If converged, it is the worst case and go to *Step 11*. Otherwise, let the new conjugate direction vector be $\mathbf{d}^{(l+1)} = \nabla G\left(\mathbf{Z}^{\mathbf{I}(k)}\right) + \beta \mathbf{d}^{(l)}$ where $\beta$ is given by $\beta = \left(\left\|\nabla G\left(\mathbf{Z}^{\mathbf{I}(k)}\right)\right\| / \left\|\nabla G\left(\mathbf{Z}^{\mathbf{I}(k-2)}\right)\right\|\right)^2$. Let $j = k$, $l = l + 1$, and go to *Step 7*.

*Step 11.* De-normalize $\mathbf{Z}^{\mathrm{I,worst}}$ by Eq. (13) in *Step 1* to obtain $\mathbf{X}^{\mathrm{I,worst}}$.

The proposed algorithm requires evaluation of sensitivities of a performance function with respect to interval variables. When gradients of the performance function are not available, sensitivities of each performance function with respect to interval variables can be calculated by the sampling-based method, and derivation of the sensitivities of the performance function with respect to interval variables and its calculation are explained in Section 3.3.

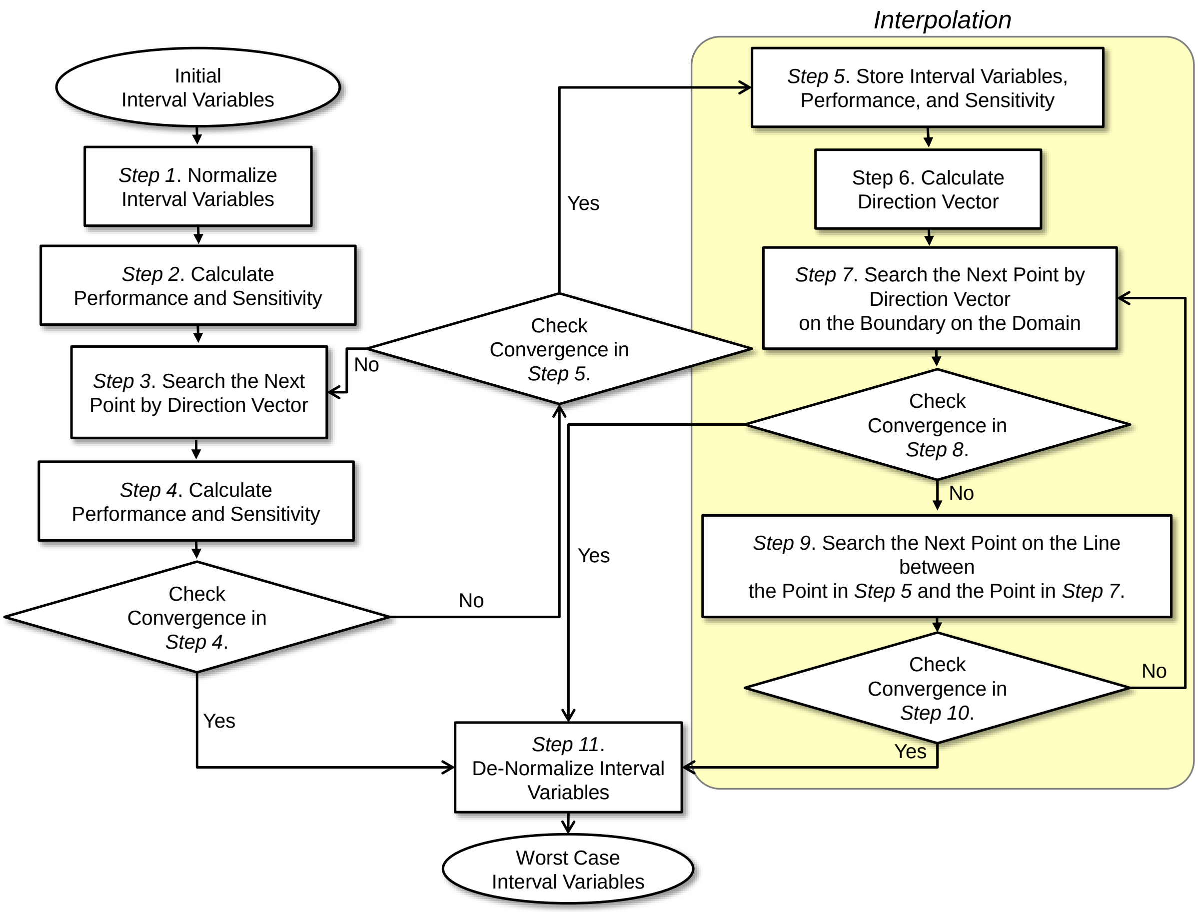


Figure 1. Flowchart for Worst Case Performance Search

**5.3 Sensitivity Analysis of Worst Case Performance Function and its Calculation**

The behavior of any point within the interval of an interval variable $x^{\mathrm{I}}$ can be expressed using the Dirac delta function $\delta_{X^{\mathrm{I}}}(\bullet)$ [45] as

$$\delta_{X^{\mathrm{I}}}\left(x^{\mathrm{I}}\right)=\begin{cases}+\infty, & x_i^{\mathrm{I}}=0\\ 0, & x_i^{\mathrm{I}}\neq 0\end{cases}, \tag{14}$$

and shifting Eq. (14) by the worst case of $x_i^{\mathrm{I}}$ denoted as $X_i^{\mathrm{I,worst}}$ yields

$$\delta_{X^{\mathrm{I}}}\left(x^{\mathrm{I}}-X^{\mathrm{I,worst}}\right)=\begin{cases}+\infty, & x^{\mathrm{I}}=X^{\mathrm{I,worst}}\\ 0, & x^{\mathrm{I}}\neq X^{\mathrm{I,worst}}\end{cases} \tag{15}$$

which is constrained to satisfy the identity

$$\int \delta_{X^{\mathrm{I}}}\left(x^{\mathrm{I}}-X^{\mathrm{I,worst}}\right)dx^{\mathrm{I}}=1. \tag{16}$$

Also, the property of the Dirac delta function [45] yields

$$\int G\left(x^{\mathrm{I}}\right)\delta_{X^{\mathrm{I}}}\left(x^{\mathrm{I}}-X^{\mathrm{I,worst}}\right)dx^{\mathrm{I}}=G\left(X^{\mathrm{I,worst}}\right). \tag{17}$$

Using Eqs. (14)~(17) and assuming $G(\bullet)$ is a continuously differentiable function of any real number, sensitivity of the worst case performance function with respect to the $i^{\text{th}}$ worst case interval variable in general dimension becomes

$$\frac{\partial G\left(\mathbf{X}_j^{\mathbf{I,worst}}\right)}{\partial X_i^{\text{I,worst}}} = \frac{\partial}{\partial X_i^{\text{I,worst}}} \int_{\mathrm{R}^{NI}} G\left(\mathbf{x}^{\mathbf{I}}\right) \delta_{\mathbf{X}^{\mathbf{I}}}\left(\mathbf{x}^{\mathbf{I}} - \mathbf{X}^{\mathbf{I,worst}}\right) d\mathbf{x}^{\mathbf{I}} = \int_{\mathrm{R}^{NI}} G\left(\mathbf{x}^{\mathbf{I}}\right) \frac{\partial \delta_{\mathbf{X}^{\mathbf{I}}}\left(\mathbf{x}^{\mathbf{I}} - \mathbf{X}^{\mathbf{I,worst}}\right)}{\partial X_i^{\text{I,worst}}} d\mathbf{x}^{\mathbf{I}} \tag{18}$$

where $\delta_{\mathbf{X}^{\mathbf{I}}}\left(\mathbf{x}^{\mathbf{I}} - \mathbf{X}^{\mathbf{I,worst}}\right) = \prod_{i=1}^{NI} \delta_{X^{\mathrm{I}}}\left(x_i^{\mathrm{I}} - X_i^{\text{I,worst}}\right)$.

Based on the definition of the Dirac delta function, behavior of a single interval variable $x^{\mathrm{I}}$ at its worst case $X^{\text{I,worst}}$ can be treated as a Gaussian normal distribution with $\mu$ of $X^{\text{I,worst}}$ and $\sigma^2$ approaching to 0, which implies

$$\delta_{X^{\mathrm{I}}}\left(x^{\mathrm{I}} - X^{\text{I,worst}}\right) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-0.5\left[\left(x^{\mathrm{I}} - X^{\text{I,worst}}\right)/\sigma\right]^2} = \lim_{\sigma \to 0} f_{X^{\mathrm{I}}}\left(x^{\mathrm{I}}\right). \tag{19}$$

Equation (19) is verified in this section first. Consider sensitivity of an one-dimensional performance function $G(\bullet)$ with respect to the worst case of interval variable $X^{\text{I,worst}}$, which by using Eq. (18) becomes

$$\frac{\partial G\left(X^{\text{I,worst}}\right)}{\partial X^{\text{I,worst}}} = \int G\left(x^{\mathrm{I}}\right) \frac{\partial \delta_{X^{\mathrm{I}}}\left(x^{\mathrm{I}} - X^{\text{I,worst}}\right)}{\partial\left(X^{\text{I,worst}}\right)} dx^{\mathrm{I}}. \tag{20}$$

$G\left(x^{\mathrm{I}}\right)$ in Eq. (20) using the Taylor series expansion at $X^{\text{I,worst}}$ can be expressed as

$$G\left(x^{\mathrm{I}}\right) = \sum_{m=0}^{\infty} \frac{G^{(m)}\left(X^{\text{I,worst}}\right)}{m!}\left(x^{\mathrm{I}} - X^{\text{I,worst}}\right)^m = \sum_{m=0}^{\infty} a_m \left(x^{\mathrm{I}} - X^{\text{I,worst}}\right)^m \tag{21}$$

where $a_m = G^{(m)}\left(X^{\text{I,worst}}\right)/m!$. Using Eqs. (19) and (21) and the score function explained in Section 2.3, the right hand side of Eq. (20) is evaluated as

$$\int G\left(x^{\mathrm{I}}\right) \frac{\partial \delta_{X^{\mathrm{I}}}\left(x^{\mathrm{I}} - X^{\text{I,worst}}\right)}{\partial X^{\text{I,worst}}} dx^{\mathrm{I}} = \int \sum_{m=0}^{\infty} a_m \left(x^{\mathrm{I}} - X^{\text{I,worst}}\right)^m \lim_{\sigma \to 0} \frac{\left(x^{\mathrm{I}} - X^{\text{I,worst}}\right)}{\sigma^2} f_{X^{\mathrm{I}}}\left(x^{\mathrm{I}}\right) dx^{\mathrm{I}} \tag{22}$$

Using the expectation operator, the Eq. (22) is further simplified as

$$\int G\left(x^{\mathrm{I}}\right) \frac{\partial \delta_{X^{\mathrm{I}}}\left(x^{\mathrm{I}} - X^{\text{I,worst}}\right)}{\partial X^{\text{I,worst}}} dx^{\mathrm{I}} = E\left[\sum_{m=0}^{\infty} \lim_{\sigma \to 0} \frac{1}{\sigma^2}\left(a_m \left(x^{\mathrm{I}} - X^{\text{I,worst}}\right)^{m+1}\right)\right] = E\left[\lim_{\sigma \to 0} \frac{1}{\sigma^2} a_1 \left(x^{\mathrm{I}} - X^{\text{I,worst}}\right)^2\right] = a_1 \tag{23}$$

where $E\left[\left(x^{\mathrm{I}} - X^{\text{I,worst}}\right)^p\right] = 0$ if $p$ is odd and $E\left[\left(x^{\mathrm{I}} - X^{\text{I,worst}}\right)^p\right] = \sigma^p (p-1)!!$ if $p$ is even according to the property of central moments of a normal distribution. Using Eq. (21), the left hand side of Eq. (20) is evaluated as

$$\frac{\partial G\left(X^{\text{I,worst}}\right)}{\partial X^{\text{I,worst}}} = \left.\frac{\partial G\left(X^{\mathrm{I}}\right)}{\partial X^{\mathrm{I}}}\right|_{X^{\mathrm{I}} = X^{\text{I,worst}}} = \left.\frac{\partial \sum_{m=0}^{\infty} a_m \left(X^{\mathrm{I}} - X^{\text{I,worst}}\right)^m}{\partial X^{\mathrm{I}}}\right|_{X^{\mathrm{I}} = X^{\text{I,worst}}} = \left.\sum_{m=1}^{\infty} a_m m \left(X^{\mathrm{I}} - X^{\text{I,worst}}\right)^{m-1}\right|_{X^{\mathrm{I}} = X^{\text{I,worst}}} = a_1. \tag{24}$$

The identical results in Eqs. (23) and (24) demonstrate the validity of treating behavior of $x^{\mathrm{I}}$ at $X^{\text{I,worst}}$ as a Gaussian normal distribution with $\mu$ of $X^{\text{I,worst}}$ and $\sigma^2$ approaching to 0.

Finally, using Eq. (19), Eq. (18) is further developed as

$$\frac{\partial G\left(\mathbf{X}^{\mathbf{I,worst}}\right)}{\partial X_i^{\mathrm{I,worst}}}=\int_{\mathrm{R}^{NI}} G\left(\mathbf{x}^{\mathbf{I}}\right)\frac{\partial \delta_{\mathbf{X}^{\mathbf{I}}}\left(\mathbf{x}^{\mathbf{I}}-\mathbf{X}^{\mathbf{I,worst}}\right)}{\partial X_i^{\mathrm{I,worst}}}d\mathbf{x}^{\mathbf{I}}=E\left[G\left(\mathbf{x}^{\mathbf{I}}\right)\lim_{\sigma\to 0}\frac{x_i^{\mathrm{I}}-X_i^{\mathrm{I,worst}}}{\sigma^2}\right] \tag{25}$$

and sensitivity of the worst case constraint with respect to design point $\overline{X_i^{\mathrm{I}}}$ in Eq. (11) becomes

$$\frac{\partial G\left(\mathbf{X}^{\mathbf{I,worst}}\right)}{\partial \overline{X_i^{\mathrm{I}}}}=\sum_{l=1}^{NI}\frac{\partial G\left(\mathbf{X}^{\mathbf{I,worst}}\right)}{\partial X_l^{\mathrm{I,worst}}}\frac{\partial X_l^{\mathrm{I,worst}}}{\partial \overline{X_i^{\mathrm{I}}}}=\sum_{l=1}^{NI}E\left[G\left(\mathbf{x}^{\mathbf{I}}\right)\lim_{\sigma\to 0}\frac{x_l^{\mathrm{I}}-X_l^{\mathrm{I,worst}}}{\sigma^2}\right]\frac{\partial X_l^{\mathrm{I,worst}}}{\partial \overline{X_i^{\mathrm{I}}}} \tag{26}$$

where

$$\begin{cases}\partial X_l^{\mathrm{I,worst}}/\partial\overline{X_i^{\mathrm{I}}}=1, & if\ \left|X_l^{\mathrm{I,worst}}-\overline{X_i^{\mathrm{I}}}\right|=\delta_l^{\mathrm{I}}/2\\ \partial X_l^{\mathrm{I,worst}}/\partial\overline{X_i^{\mathrm{I}}}=0, & if\ \left|X_l^{\mathrm{I,worst}}-\overline{X_i^{\mathrm{I}}}\right|<\delta_l^{\mathrm{I}}/2\end{cases}. \tag{27}$$

Additionally, it is noted that the Dirac delta function can be also applicable to define behavior of deterministic variables when sensitivities of performance functions with respect to deterministic variables are not available. Thus, in the presence of deterministic variables, the proposed sampling-based method can be applied to evaluate sensitivities of performance functions even when gradients of the performance functions are not obtainable. Denote a surrogate model for the constraint function with interval variables as $\hat{G}\left(\mathbf{X}^{\mathbf{I}}\right)$. The MCS can be applied to calculate sensitivity of a performance function with respect to the $i^{\text{th}}$ worst case interval variable during the worst case performance search in Section 3.2 using Eq. (25) as

$$\frac{\partial G\left(\mathbf{X}^{\mathbf{I,worst}}\right)}{\partial X_l^{\mathrm{I,worst}}}\cong\frac{\partial \hat{G}\left(\mathbf{X}^{\mathbf{I,worst}}\right)}{\partial X_l^{\mathrm{I,worst}}}\cong\frac{1}{K}\sum_{k=1}^{K}\hat{G}\left(\mathbf{x}^{\mathbf{I,worst}(k)}\right)\frac{x_l^{\mathrm{I}(k)}-X_l^{\mathrm{I,worst}}}{\sigma^2_{X_l^{\mathrm{I,worst}}}} \tag{28}$$

where $\sigma_{X_l^{\mathrm{I,worst}}}$ is $\sigma_{X^{\mathrm{I}}}$ for $X_l^{\mathrm{I,worst}}$ coming from $\sigma$ in Eq. (26), and sensitivities of the worst case constraints in Eq. (11) can be calculated using Eqs. (26) and (27) as

$$\frac{\partial G_j\left(\mathbf{X}_j^{\mathbf{I,worst}}\right)}{\partial \overline{X_i^{\mathrm{I}}}}\cong\frac{\partial \hat{G}_j\left(\mathbf{X}_j^{\mathbf{I,worst}}\right)}{\partial \overline{X_i^{\mathrm{I}}}}\cong\frac{1}{K}\sum_{l=1}^{NI}\sum_{k=1}^{K}\hat{G}_j\left(\mathbf{x}_j^{\mathbf{I,worst}(k)}\right)\frac{x_{j,l}^{\mathrm{I}(k)}-X_{j,l}^{\mathrm{I,worst}}}{\sigma^2_{X_{j,l}^{\mathrm{I,worst}}}}\frac{\partial X_{j,l}^{\mathrm{I,worst}}}{\partial \overline{X_{j,i}^{\mathrm{I}}}}. \tag{29}$$

The desired value of $\sigma_{X^{\mathrm{I}}}$ used in Eqs. (28) and (29) for the sampling-based method is determined through the following simulation analysis. During the simulation analysis, ratio of $\sigma_{X^{\mathrm{I}}}$ to $\overline{X^{\mathrm{I}}}$ or $\frac{\sigma_{X^{\mathrm{I}}}}{\overline{X^{\mathrm{I}}}}$ instead of just $\sigma_{X^{\mathrm{I}}}$ is considered since $\sigma_{X^{\mathrm{I}}}$ depends on $\overline{X^{\mathrm{I}}}$, and sensitivity of a performance function $G_1\left(X^{\mathrm{I}}\right)=2X^{\mathrm{I}}$ with respect to $X^{\mathrm{I}}$ is calculated by the MCS while $\frac{\sigma_{X^{\mathrm{I}}}}{\overline{X^{\mathrm{I}}}}$ changes from 0.1 to 0.001 in descending order. The result is then compared to the true sensitivity, which is analytically obtained as 2. From the result shown in Figure 2, it is demonstrated that $0.008\le\frac{\sigma_{X^{\mathrm{I}}}}{\overline{X^{\mathrm{I}}}}\le 0.02$ for the desired value of $\sigma_{X^{\mathrm{I}}}$. From the negligible amount of error less than 0.3% in Figure 2, validity of calculating sensitivities of a performance function with respect to interval variables using the sampling-based method with a very small standard deviation can be also shown.

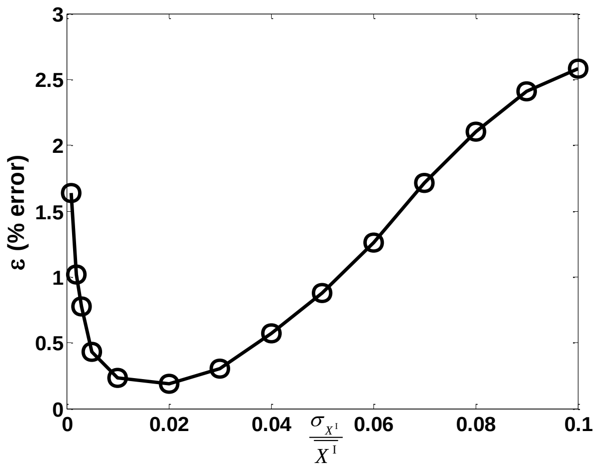


Figure 2. Error of Sensitivity as $\frac{\sigma_{X^{\mathrm{I}}}}{\overline{X^{\mathrm{I}}}}$ changes from 0.001 to 0.1

## 6. Design Optimization with Random and Interval Variables

### 6.1 Formulation of Design Optimization with Mixture of Random and Interval Variables

The mathematical formulation of design optimization with mixture of random and interval variables is expressed as

$$\begin{aligned} &\text{minimize} && \text{cost}(\mathbf{d}) \\ &\text{subject to} && P_{F_j}^{\text{worst}} \equiv P\left[G_j\left(\mathbf{X}^{\mathbf{R}}, \mathbf{X}_j^{\mathbf{I,worst}}\right) > 0\right] \le P_{F_j}^{\text{tar}}, \quad j = 1,\dots,\text{NC} \\ &&& \mathbf{d}^{\text{L}} \le \mathbf{d} \le \mathbf{d}^{\text{U}}, \quad \mathbf{d} \in \text{R}^{\text{ndv}}, \quad \mathbf{X}^{\mathbf{R}} \in \text{R}^{NR}, \text{ and } \mathbf{X}^{\mathbf{I}} \in \text{R}^{NI} \end{aligned} \tag{30}$$

where $\mathbf{d} = \left\{\mu_1,\dots,\mu_{NR}, \overline{X_{NR+1}^{\text{I}}},\dots,\overline{X_{NR+NI}^{\text{I}}}\right\}$ is the design vector; $\mathbf{X}_j^{\mathbf{I,worst}}$ in Eq. (30) is the worst case interval variables for the $j^{\text{th}}$ probabilistic constraint, which is obtained by solving the optimization problem to

$$\begin{aligned} &\text{maximize} && P\left[G_j\left(\mathbf{X}^{\mathbf{I}}\right) > 0\right] \\ &\text{subject to} && \left|X_i^{\text{I}} - \overline{X_i^{\text{I}}}\right| \le \frac{\delta_i^{\text{I}}}{2} \ \ for\ \ i = 1,\dots,NI. \end{aligned} \tag{31}$$

To carry out the design optimization with interval variables using Eq. (30), probabilistic constraints with the worst case interval variables, namely the worst case probabilistic constraints or the worst case probability of failure, and their sensitivities must be evaluated. Each of the worst case probabilistic constraint is obtained by the worst case probability of failure search that solves Eq. (31) and will be explained in Section 4.2, and sensitivity analysis of each of the worst case probabilistic constraint and its calculation are explained in Section 4.3. It should be noted that the worst case probability of failure does not always occur at the point where the worst case performance occurs, which is demonstrated with an example in Section 4.2. Thus, by applying an algorithm for the worst case performance search in Section 3.2 by directly utilizing probability of failure and its sensitivity in replacement of performance value and its sensitivity, the problem pointed out in the previous sentence can be resolved.

### 6.2 Worst Case Probability of Failure

The worst case probability of failure with random and interval variables, denoted by $P_F^{\text{worst}}$, is defined using Eq. (30) and a multi-dimensional integral as

$$P_F^{\mathbf{worst}} \equiv \int_{\text{R}^{NR+NI}} I_{\Omega_F}\left(\mathbf{x}^{\mathbf{R}}, \mathbf{x}^{\mathbf{I}}\right) f_{\mathbf{X}}\left(\mathbf{x}^{\mathbf{R}}\right) \delta_{\mathbf{X}^{\mathbf{I}}}\left(\mathbf{x}^{\mathbf{I}} - \mathbf{X}^{\mathbf{I,worst}}\right) d\mathbf{x}^{\mathbf{R}} d\mathbf{x}^{\mathbf{I}} = E\left[I_{\Omega_F}\left(\mathbf{X}^{\mathbf{R}}, \mathbf{X}^{\mathbf{I,worst}}\right)\right]. \tag{32}$$

The worst case probability of failure in Eq. (32) is obtained using the algorithm for the worst case performance search explained in Section 3.2 by utilizing probability of failure and its sensitivity in replacement of the performance function and its sensitivity. Derivation of the sensitivity of the probability of failure with respect to the worst case interval variables and its calculation are explained in Section 4.3. Usually, the worst case probability of failure occurs at the worst case performance, so conventionally the worst case probability of failure is calculated by evaluating the probability of failure at the worst case performance [15-17]. However, this is not always the case and the following example demonstrates it.

Consider a 2D highly nonlinear polynomial function,

$$G_2(\mathbf{X}) = 0.7361 + (\mathrm{W}-6)^2 + (\mathrm{W}-6)^3 - 0.6\times(\mathrm{W}-6)^4 + \mathrm{Z} \tag{33}$$

Table 1. Property of Input Variables

| Variables | Types | Distribution | Parameters | |
|---|---|---|---|---|
| $X_1^{\mathrm{I}}$ | Interval | N/A | $\overline{X_1^{\mathrm{I}}} = 6.5$ | $\delta_1^{\mathrm{I}} = 3$ |
| $X_2^{\mathrm{R}}$ | Random | Normal | $\mu_2 = 2.5$ | $\sigma_2 = 1$ |

where $\begin{Bmatrix} \mathrm{W} \\ \mathrm{Z} \end{Bmatrix} = \begin{bmatrix} 0.8660 & 0.5000 \\ 0.5000 & -0.8660 \end{bmatrix} \begin{Bmatrix} X_1^{\mathrm{I}} \\ X_2^{\mathrm{R}} \end{Bmatrix}$. As shown in Table 1, $X_1^{\mathrm{I}}$ and $X_2^{\mathrm{R}}$ are interval and random variables, respectively. The mid-point and interval length of $X_1^{\mathrm{I}}$ are 6.5 and 3, respectively. The mean and standard deviation of $X_2^{\mathrm{R}}$ are 2.5 and 1, respectively. Then, $X_1^{\mathrm{I}}$ is divided into 100 sub-intervals, for each of which, the performance functions and probability of failures are evaluated. For the evaluation of the probability of failure, $5\times10^7$ MCS sample are used for each sub-interval.

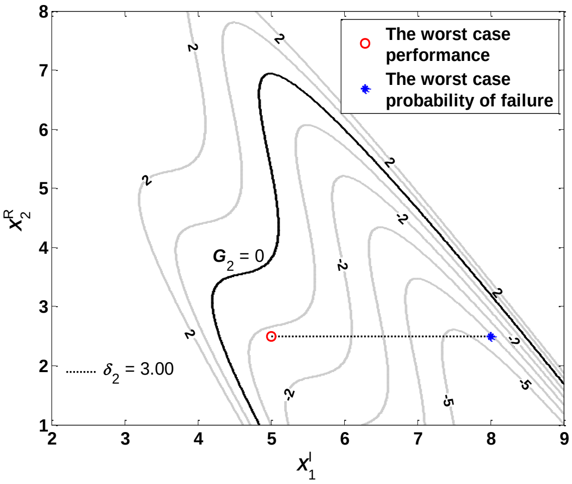


Figure 3. Worst Case Performance and Worst Case Probability of Failure

As shown in Fig. 3, the worst case probability of failure does not occur where the worst case performance occurs. The worst case probability of failure occurs at $X_1^{\mathrm{I}} = \overline{X_1^{\mathrm{I}}} + \delta_1^{\mathrm{I}}/2 = 8$ where performance and probability of failure are −4.1437 and 0.2418, respectively, while the worst case performance occurs at $X_1^{\mathrm{I}} = \overline{X_1^{\mathrm{I}}} - \dfrac{\delta_1^{\mathrm{I}}}{2} = 5$, where the performance and probability of failure are −1.1547 and 0.1222, respectively. Thus, this study suggests using the algorithm for the worst case probability failure search directly instead of obtaining the worst case probability of failure by calculating the probability of failure where the worst case performance occurs.

The MCS can be applied to calculate probability of failure during the worst case probability of failure search. Denoting the surrogate model for constraint functions with random and interval variables as $\hat{G}\left(\mathbf{X^R}, \mathbf{X^I}\right)$, the probability of failure during the worst case probability of failure search can be calculated using Eq. (32) as

$$P_F^{\mathrm{worst}} \equiv P\left[G\left(\mathbf{X^R}, \mathbf{X^{I,worst}}\right) > 0\right] \cong \frac{1}{K}\sum_{k=1}^{K} I_{\hat{\Omega}_F}\left[\mathbf{X}^{\mathbf{R}(k)}, \mathbf{X^{I,worst}}\right] \le P_F^{\mathrm{tar}} \tag{34}$$

where the failure set $\hat{\Omega}_F$ for the surrogate model is defined as $\hat{\Omega}_F \equiv \left[\mathbf{x} : \hat{G}\left(\mathbf{x}^{\mathbf{R}}, \mathbf{X}^{\mathbf{I,worst}}\right) > 0\right]$.

**6.3 Sensitivity Analysis of Worst Case Probability of Failure and its Calculation**

Taking partial derivative of Eq. (32) with respect to the $i^{\text{th}}$ worst case interval variable yields

$$\frac{\partial P_F^{\text{worst}}}{\partial X_i^{\text{I,worst}}} = \sum_{i=1}^{NI} E\left[ I_{\Omega_F}\left(\mathbf{x}^{\mathbf{R}}, \mathbf{X}^{\mathbf{I,worst}}\right) \frac{x_i^{\text{I}} - X_i^{\text{I}}}{\sigma^2_{X_i^{\text{I,worst}}}} \right]. \tag{35}$$

Then, taking partial derivative of Eq. (32) with respect to the mid-point of the $i^{\text{th}}$ interval variable using Eq. (35) yields

$$\frac{\partial P_F^{\text{worst}}}{\partial \overline{X_i^{\text{I}}}} = \sum_{l=1}^{NI} \frac{\partial P_F^{\text{worst}}}{\partial X_l^{\text{I,worst}}} \frac{\partial X_l^{\text{I,worst}}}{\partial \overline{X_i^{\text{I}}}} = \sum_{l=1}^{NI} E\left[ I_{\Omega_F}\left(\mathbf{x}^{\mathbf{R}}, \mathbf{X}^{\mathbf{I,worst}}\right) \frac{x_l^{\text{I}} - X_l^{\text{I}}}{\sigma^2_{X_l^{\text{I,worst}}}} \right] \frac{\partial X_l^{\text{I,worst}}}{\partial \overline{X_i^{\text{I}}}}, \tag{36}$$

where $\frac{\partial X_l^{\text{I,worst}}}{\partial \overline{X_i^{\text{I}}}}$ is obtained from Eq. (27). Taking partial derivative of Eq. (32) with respect to the mean of the $i^{\text{th}}$ random variable yields

$$\frac{\partial P_F^{\mathbf{worst}}}{\partial \mu_i} = \int_{\mathrm{R}^{NR+NI}} I_{\Omega_F}\left(\mathbf{x}^{\mathbf{R}}, \mathbf{x}^{\mathbf{I}}\right) \frac{\partial \ln f_{\mathrm{X}}\left(\mathbf{x}^{\mathbf{R}}; \boldsymbol{\mu}\right)}{\partial \mu_i} f_{\mathrm{X}}\left(\mathbf{x}^{\mathbf{R}}; \boldsymbol{\mu}\right) \delta_{\mathrm{X}^{\mathrm{I}}}\left(\mathbf{x}^{\mathbf{I}} - \mathbf{X}^{\mathbf{I,worst}}\right) d\mathbf{x}^{\mathbf{R}} d\mathbf{x}^{\mathbf{I}}. \tag{37}$$

Using Eq. (17), Eq. (37) is further simplified as

$$\frac{\partial P_F^{\text{worst}}}{\partial \mu_i} = \int_{\mathrm{R}^{NR+NI}} I_{\Omega_F}\left(\mathbf{x}^{\mathbf{R}}, \mathbf{X}^{\mathbf{I,worst}}\right) \frac{\partial \ln f_{\mathrm{X}}\left(\mathbf{x}^{\mathbf{R}}; \boldsymbol{\mu}\right)}{\partial \mu_i} f_{\mathrm{X}}\left(\mathbf{x}^{\mathbf{R}}; \boldsymbol{\mu}\right) d\mathbf{x}^{\mathbf{R}} = E\left[ I_{\Omega_F}\left(\mathbf{x}^{\mathbf{R}}, \mathbf{X}^{\mathbf{I,worst}}\right) \frac{\partial \ln f_{\mathrm{X}}\left(\mathbf{x}^{\mathbf{R}}; \boldsymbol{\mu}\right)}{\partial \mu_i} \right]. \tag{38}$$

The MCS can be applied to calculate sensitivity of the probability of failure with respect to the $i^{\text{th}}$ worst case interval variable during the worst case probability of failure search in Section 4.2 based on Eq. (35) as

$$\frac{\partial P_F^{\text{worst}}}{\partial X_i^{\text{I,worst}}} = \frac{1}{K} \sum_{i=1}^{K} I_{\hat{\Omega}_F}\left[\mathbf{x}^{\mathbf{R}(k)}, \mathbf{X}^{\mathbf{I,worst}}\right] \frac{x_i^{\text{I}(k)} - X_i^{\text{I,worst}}}{\sigma^2_{X_i^{\text{I,worst}}}}. \tag{39}$$

Sensitivities of the worst case probabilistic constraints in Eq. (30) with respect to the $i^{\text{th}}$ interval variable at the mid-point as

$$\frac{\partial P_{F_j}^{\text{worst}}}{\partial \overline{X_i^{\text{I}}}} \cong \frac{1}{K} \sum_{l=1}^{NI} \sum_{k=1}^{K} I_{\hat{\Omega}_{F_j}}\left[\mathbf{x}^{\mathbf{R}(k)}, \mathbf{X}_{\boldsymbol{j}}^{\mathbf{I,worst}}\right] \frac{x_{j,l}^{\text{I}(k)} - X_{j,l}^{\text{I,worst}}}{\sigma^2_{X_{j,l}^{\text{I,worst}}}} \frac{\partial X_{j,l}^{\text{I,worst}}}{\partial \overline{X_{j,i}^{\text{I}}}} \tag{40}$$

based on Eq. (36). Sensitivities of the worst case probabilistic constraints in Eq. (30) with respect to the $i^{\text{th}}$ random variable at the mean point are calculated as

$$\frac{\partial P_{F_j}^{\text{worst}}}{\partial \mu_i} \cong \frac{1}{K} \sum_{k=1}^{K} I_{\hat{\Omega}_{F_j}}\left[\mathbf{x}^{\mathbf{R}(k)}, \mathbf{X}_{\boldsymbol{j}}^{\mathbf{I,worst}}\right] s_{\mu_i}^{(1)}\left[\mathbf{x}^{\mathbf{R}(k)}; \boldsymbol{\mu}\right] \tag{41}$$

based on Eq. (38).

**7. Numerical Examples**

Numerical studies are carried out in this section to verify the algorithm that searches the worst case probability of failure in Section 4 for both low-dimensional and high-dimensional cases. Also, design optimization with mixture of interval and random variables that utilizes the worst case probability of failure search is carried out.

### 7.1 Worst Case Probability of Failure Search for Two-Dimensional Inputs

In this numerical example, the algorithm that searches the worst case probability of failure is applied to a two-dimensional case, and one of input variables is an interval and the other is a random. Consider a nonlinear performance function given as

$$G_1(\mathbf{X}) = -0.3X_1^{\mathrm{I}}\left(X_2^{\mathrm{R}}\right)^2 + X_1^{\mathrm{I}} - 0.8X_2^{\mathrm{R}} - 2.8. \tag{42}$$

Table 2. Property of Input Variables

| Variables | Types | Distribution | Parameters | |
|---|---|---|---|---|
| $X_1^{\mathrm{I}}$ | Interval | NA | $\overline{X_1^{\mathrm{I}}} = -0.5$ | $\delta_1^{\mathrm{I}} = 1$ |
| $X_2^{\mathrm{R}}$ | Random | Normal | $\mu_2 = 2.2$ | $\sigma_2 = 1$ |

As shown in Table 2, $X_1^{\mathrm{I}}$ is an interval variable with its mid-point at −0.5 and its interval length of 1, and $X_2^{\mathrm{R}}$ is a normally distributed random variable with its mean at 2.2 and its standard deviation of 1.

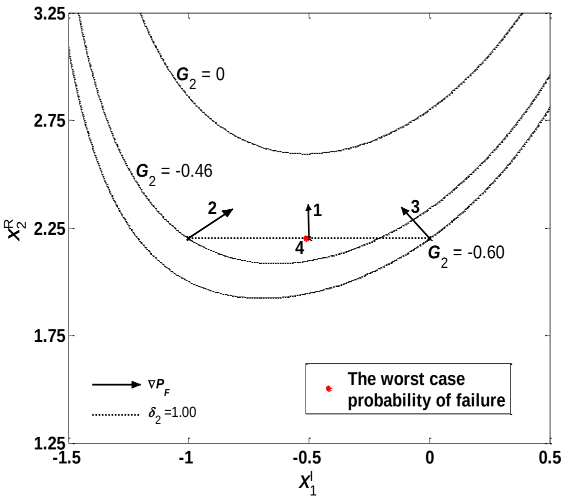


Figure 4. Search History of Worst Case Probability of Failure

Table 3. Search History of Worst Case Probability of Failure

| Iteration | $X_1^{\mathrm{I}}$ | $P_{F_1}$ | $\partial P_{F_1} / \partial X_1^{\mathrm{I}}$ |
|---|---|---|---|
| 1 | −0.5000 | 0.3457 | -0.00789 |
| 2 | −1.0000 | 0.2541 | 0.42073 |
| 3 | 0.0000 | 0.2737 | -0.26648 |
| **4** | **−0.5136** | **0.3460** | **0.00082** |

With the given property of these input variables and the performance function in Eq. (42), the worst case probability of failure is obtained using the worst case probability of failure search explained in Section 4.2. The results are shown both in Table 3 and Fig. 4. The worst case interval variables at the 4th iteration in Table 3 is obtained by an interpolation of two worst case interval variables candidates at the 2nd and the 3rd iteration during *Step 9* of the worst case probability of failure search. The obtained result is compared with the result obtained by dividing the interval domain into 100 sub-intervals and performing the MCS with $5\times10^7$ samples for all sub-intervals, which is shown in Table 4.

Table 4. Comparison of Results Obtained by 2 Different Methods

| Methods | $X_1^{\mathrm{I}}$ | $P_{F_1}^{\mathrm{worst}}$ | Number of MCS |
|---|---|---|---|
| Proposed Algorithm | -0.5136 | 0.3460 | 4 |
| Performing MCS for all 100 sub-intervals | -0.5152 | 0.3465 | 100 |

In terms of efficiency, the proposed algorithm requires (4iterations) × (1MCS/iteration) = 4MCSs, and performing the MCS for all 100 sub-divided intervals requires (100sub-intervals) × (1MCS/sub-interval) = 100MCSs. Thus, the proposed algorithm is 25 times more efficient than the crude MCS while maintaining accuracy in this example.

**7.2 Worst Case Probability of Failure Search for High-Dimensional Inputs**

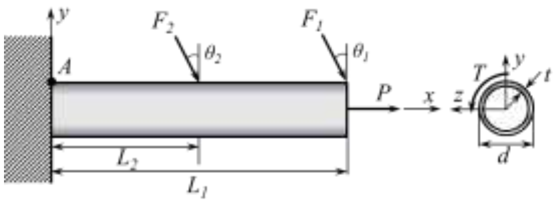


Figure 5. Schematic Diagram of Cantilever Tube

In this numerical example, the algorithm that searches the worst case probability of failure is applied to a high-dimensional case where 2 of input variables are interval and 9 of them are random variables. Consider the cantilever tube shown in Fig. 5 subjected to external forces $F_1$, $F_2$, and $P$, and torsion $T$ [16]. The performance function is defined as the difference between the yield strength $S_y$ and the maximum stress $\sigma_y$ , namely,

$$G_2 = g(\mathbf{X}) = \sigma_{\max} - S_y \tag{43}$$

where $\sigma_{\max}$ is the maximum von Mises stress on the top surface of the tube at the origin, which is given by

$$\sigma_{\max} = \sqrt{\sigma_x^2 + 3\tau_{zx}^2} \tag{44}$$

where the normal stress $\sigma_x$ is obtained

$$\sigma_x = \frac{P + F_1 \sin\theta_1 + F_2 \sin\theta_2}{\frac{\pi}{4}\left[d^2 - (d-2t)^2\right]} + \frac{\left(F_1 L_1 \cos\theta_1 + F_2 L_2 \cos\theta_2\right)d}{2 \times \frac{\pi}{64}\left[d^4 - (d-2t)^4\right]} \tag{45}$$

and the shear stress $\tau_{xz}$ is obtained as

$$\tau_{xz} = \frac{Td}{4 \times \frac{\pi}{64}\left[d^4 - (d-2t)^4\right]}, \tag{46}$$

respectively. The property of random and interval variables are given in Tables 5 and 6, respectively. As shown in Tables 5 and 6, nine random variables $X_1^{\mathrm{R}} \sim X_9^{\mathrm{R}}$ having various distributions and two interval variables $X_{10}^{\mathrm{I}}$ and $X_{11}^{\mathrm{I}}$ having the identical interval length at different mid-points are used as input variables.
With the property of input variables and the performance function in Eq. (43), the worst case probability of failure is obtained using the worst case probability of failure search. The MCS with $5 \times 10^7$ samples is tried for every iteration, and the tolerance of $10^{-4}$ instead of $10^{-3}$ is set for this example since the sensitivity of probability of failure with respect to both interval variables is less than $10^{-2}$ throughout the interval domain. By using the

Table 5. Property of Random Variables

| Variables | Parameter 1 | Parameter 2 | Distribution |
|---|---|---|---|
| $X_1^{\mathrm{R}}(t)$ | 5 mm (mean) | 0.1 mm (std*) | Normal |
| $X_2^{\mathrm{R}}(d)$ | 42 mm (mean) | 42 mm (mean) | Normal |
| $X_3^{\mathrm{R}}(L_1)$ | 119.75 mm (lb**) | 120.25 mm (ub***) | Uniform |
| $X_4^{\mathrm{R}}(L_2)$ | 59.75 mm (lb) | 60.25 mm (ub) | Uniform |
| $X_5^{\mathrm{R}}(F_1)$ | 3.0 kN (mean) | 0.3 kN (std) | Normal |
| $X_6^{\mathrm{R}}(F_2)$ | 3.0 kN (mean) | 0.3 kN (std) | Normal |
| $X_7^{\mathrm{R}}(P)$ | 12.0 kN (mean) | 1.2 kN (std) | Gumbel |
| $X_8^{\mathrm{R}}(T)$ | 90.0 $\mathrm{N \cdot m}$ (mean) | 9.0 $\mathrm{N \cdot m}$ (std) | Normal |
| $X_9^{\mathrm{R}}(S_y)$ | 133.7 MPa (mean) | 22.0 MPa (std) | Normal |

***:** std-standard deviation
****:** lb – lower bound of a uniform distribution
*****:** ub – upper bound of a uniform distribution

Table 6. Property of Interval Variables

| Variables | Parameters |
|---|---|
| $X_{10}^{\mathrm{I}}(\theta_1)$ | $\overline{X}_{10}^{\mathrm{I}} = 5°$ , $\delta_{10}^{\mathrm{I}} = 10°$ |
| $X_{11}^{\mathrm{I}}(\theta_2)$ | $\overline{X}_{11}^{\mathrm{I}} = 10°$ , $\delta_{11}^{\mathrm{I}} = 10°$ |

proposed algorithm, the worst case probability of failure is obtained in 8 iterations including the one with the interpolation and the discard one. In Table 7, since the probability of failure at the 4$^{\text{th}}$ iteration is smaller than that at the 3$^{\text{rd}}$ iteration, it is discarded during the *Step 5* of the worst case probability of failure search in Section 4.2. Search history is shown in both Table 7 and Fig. 6. The worst case probability of failure is obtained as 0.50849 and the worst case interval variables are obtained as [3.993, 7.887]. The obtained result is then compared with the result obtained by dividing both interval domains into 100 sub-intervals and performing MCS with $5\times10^7$ samples for all combinations of sub-intervals.

Table 7. Search History of Worst Case Probability of Failure

| Iteration | $X_{10}^{\mathrm{I}}(\theta_1)$ | $X_{11}^{\mathrm{I}}(\theta_2)$ | $P_{F_2}$ | $\partial P_{F_2} / \partial X_{10}^{\mathrm{I}}$ | $\partial P_{F_2} / \partial X_{11}^{\mathrm{I}}$ |
|---|---|---|---|---|---|
| 1 | 5.000 | 10.00 | 0.50788 | -3.940E-04 | -4.0768E-04 |
| 2 | 0.000 | 5.000 | 0.50476 | 1.505E-03 | 5.479E-04 |
| 3 | 0.000 | 5.000 | 0.50478 | 1.486E-03 | 5.205E-04 |
| 4* | 10.00 | 15.00 | 0.49676 | -2.301E-03 | -1.345E-03 |
| 3' | 10.00 | 8.502 | 0.50160 | -2.249E-03 | -1.190E-04 |
| 4 | 4.189 | 6.467 | 0.50828 | -5.640E-05 | 2.800E-04 |
| 4' | 3.015 | 15.00 | 0.50343 | 3.712E-04 | 1.362E-03 |
| **5** | **3.993** | **7.887** | **0.50849** | **-2.790E-05** | **2.1116E-07** |

*: Discarded during *Step 5* of Worst Case Probability of Failure Search
': Utilized for Interpolation

The result of the comparison is shown in Table 8. In terms of efficiency, the proposed algorithm requires (8iterations) × (1MCS/iteration) = 8MCSs, and performing MCS for all combinations of 100 sub-intervals requires (100×100combinations) × (1MCS/combination) = 10000MCSs. Thus, the proposed algorithm is 1250 times more efficient while maintaining accuracy in this example. As suggested by the current and the previous examples, the more interval variables there are, the less efficient performing the crude MCS exponentially becomes. In general dimension, performing the MCS for all combinations of 100 sub-intervals of every interval variable requires ($100^{NI}$

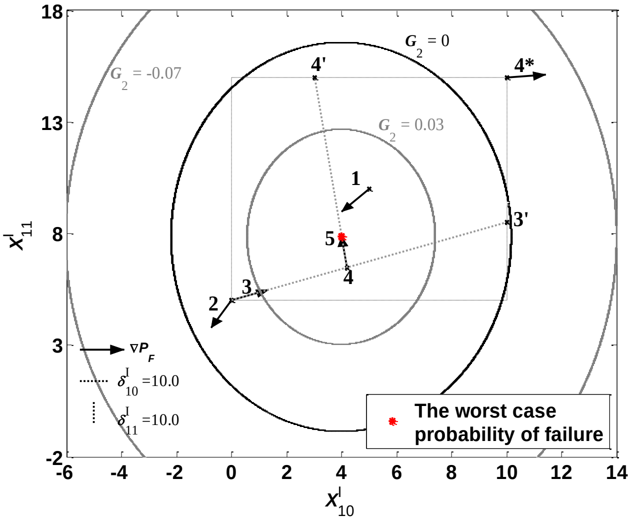

Figure 6. Search History of Worst Case Probability of Failure

combinations) × (1MCS/combination) = $(10)^{2NI}$ MCSs. On the other hand, the proposed algorithm requires similar number of MCSs regardless of dimension of interval variables since it utilizes a vector of interval variables and its sensitivity vector instead of their individual components.

Table 8. Comparison of Results Obtained by 2 Different Methods

| Methods | Worst Case Interval Variable | Probability of Failure | Number of MCSs |
|---|---|---|---|
| Proposed Algorithm | [3.993 7.887] | 0.50849 | 8 |
| Performing MCS for all combinations of 100 sub-intervals | [3.939 7.879] | 0.50850 | 10000 |

### 7.3 Design Optimization with Mixture of Random and Interval Inputs

This numerical example shows the design optimization with mixture of random and interval variables, utilizing the worst case probability of failure search. Consider a 2D mathematical design optimization problem, which is formulated to

$$\begin{aligned} &\text{minimize} \quad C(\mathbf{d}) = d_1 + d_2 \\ &\text{subject to} \quad P_{F_j}^{\text{worst}} = P\left[G_j\left(\mathbf{X}^{\mathbf{R}}(\mathbf{d}), \mathbf{X}_j^{\mathbf{I},\text{worst}}\right) > 0\right] \le P_{F_j}^{\text{tar}} = 2.275\%, \quad j = 1 \sim 3 \\ &\mathbf{d}^{\text{L}} \le \mathbf{d} \le \mathbf{d}^{\text{U}}, \quad \mathbf{d} \in \text{R}^2, \quad \mathbf{X}^{\mathbf{R}} \in \text{R}, \quad \text{and} \quad \mathbf{X}^{\mathbf{I}} \in \text{R} \end{aligned} \tag{47}$$

where three constraints are given by

$$\begin{aligned} G_1(\mathbf{X}) &= 1 - \frac{\left(X_1^{\text{I}}\right)^2 X_2^{\text{R}}}{20} \\ G_2(\mathbf{X}) &= 1 - \frac{\left(X_1^{\text{I}} + X_2^{\text{R}} - 5\right)^2}{30} - \frac{\left(X_1^{\text{I}} - X_2^{\text{R}} - 12\right)^2}{120} \\ G_3(\mathbf{X}) &= 1 - \frac{80}{\left(X_1^{\text{I}}\right)^2 + 8X_2^{\text{R}} + 5}. \end{aligned} \tag{48}$$

The properties of two input variables, one interval and one random variable, are shown in Table 9. As shown in Eq. (47), the target probability of failure $\left(P_{F_j}^{\text{tar}}\right)$ is set to 2.275% for all constraints.

Fig. 7 shows the optimum design of the sampling-based design optimization with interval and random variables. As can be seen in Fig. 7, the deterministic design optimum ($\text{d}^{\text{dopt}}$) was first searched to enhance efficiency of the

Table 9. Property of Input Variables

| Input Variables | Variable Types | $d^L$ | $d^O$ | $d^U$ | Parameters |
|---|---|---|---|---|---|
| $X_1^I$ | Interval | 0.0 | 5.0 | 10.0 | $\delta_1^I = 1.2$ |
| $X_2^R$ | Random | 0.0 | 5.0 | 10.0 | $\sigma_2 = 0.4$ |

design optimization procedure. In Fig. 7, the dotted box illustrated around the design optimum ($d^{opt}$) shows the joint range of $X_1^I$ and $X_2^R$. With $P_{F_j}^{tar}$ of 2.275%, allowed total range of distribution of $X_2^R$ becomes $4\times\sigma_2 = 1.6$, and with $\delta_1^I = 1.2$ of $X_1^I$, size of the dotted box becomes 1.2×1.6. With the dotted box around $d^{opt}$ it is easily identified that $d^{opt}$ is the desired optimum as vertices of the box are right on two active constraints, $G_1(\mathbf{X}) = 0$ and $G_2(\mathbf{X}) = 0$. $P_{F_1}^{worst}$ and $P_{F_2}^{worst}$ occur on the left and the right bounds of $X_1^I$, respectively where $P_{F_1}^{worst}$ and $P_{F_2}^{worst}$ are 0.0231 and 0.0228, respectively, which are very close to $P_{F_j}^{tar}$. Design search history and number of iterations taken to obtain the worst case probability of failure at each design are shown in Table 10. One MCS for each iteration is used to obtain $P_{F_j}^{worst}$ and $\nabla P_{F_j}^{worst}$ while applying the worst case probability of failure search explained in Section 4.2. At the 2nd iteration during the design search, $P_{F_2}$ does not behave monotonic within the domain of the interval variable, thus the interpolation algorithm is applied to find more accurate worst case probability of failure, which is why 5 iterations are taken to obtain $P_{F_2}^{worst}$. Overall iterations taken to obtain $P_{F_j}^{worst}$ is around 2 for each design search since $P_{F_2}$ behaves monotonic most times within the domain of the interval variable. Thus it is concluded

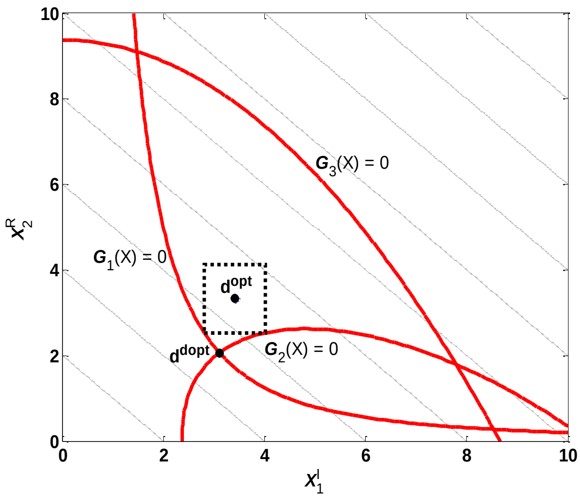


Figure 7. Optimum Design of Sampling-Based Design Optimization with Random and Interval Variables

that the computational burden to obtain $P_{F_j}^{worst}$ in this example is affordable.

Table 10. Design Search History and Number of Iterations for Worst Case Probability of Failure

| Iteration | Design Point ($d_1$, $d_2$) | # Iterations for $P_{F_1}^{worst}$ | # Iterations for $P_{F_2}^{worst}$ | # Iterations for $P_{F_3}^{worst}$ |
|---|---|---|---|---|
| 1 | (3.1139, 2.0639) | 2 | 2 | 2 |
| 2 | (4.3814, 3.6368) | 2 | **5** | 2 |
| 3 | (3.8142, 2.9765) | 2 | 2 | 2 |
| 4 | (2.8478, 3.0296) | 2 | 2 | 2 |
| 5 | (3.1377, 3.0137) | 2 | 2 | 2 |
| 6 | (3.3407, 3.0025) | 2 | 2 | 2 |
| 7 | (3.3741, 3.1111) | 2 | 2 | 2 |
| 8 | (3.3838, 3.1274) | 2 | 2 | 2 |

## 8. Conclusions

Sampling-based design optimizations with only interval variables and with both interval and random variables are developed in this study. For the design optimization with interval variables only, each of the worst case constraint is evaluated by the developed worst case performance search where interval and sensitivity vector are utilized, thus efficiency is promised regardless of the dimension of the interval variables. It is assumed that gradients of performance functions are not available in this study. Therefore, sensitivities of a performance function with respect to interval variables are derived by defining behavior of the interval variables at the worst case by the Dirac delta function to calculate it by the sampling-based method. Through the simulation analysis, desired value of standard deviation for the interval variables at the worst case is determined, and the error of the result turns out to be negligible at the desired value. Using the obtained value, the sensitivities of each of the worst case constraints both at the worst case and design points are calculated by the MCS. For the design optimization with random and interval variables, the worst case probabilistic constraints are evaluated by the worst case probability of failure search. Since probability of failure does not always occur where the worst case performance occurs as demonstrated in this study, the worst case probability of failure is obtained by directly using the probability of failure and its sensitivity. Similarly to design optimization with interval variables only, the sensitivities of the probability of failure both at the worst case and design points are derived, which are then calculated by the MCS. Numerical examples show the worst case probability of failure is obtained efficiently for both low and high-dimensional inputs regardless of the dimension of the interval variables and the design optimization with random and interval variables is successfully carried out with efficiency utilizing the worst case probability of failure search.

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