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Shape Optimization for Brake Squeal

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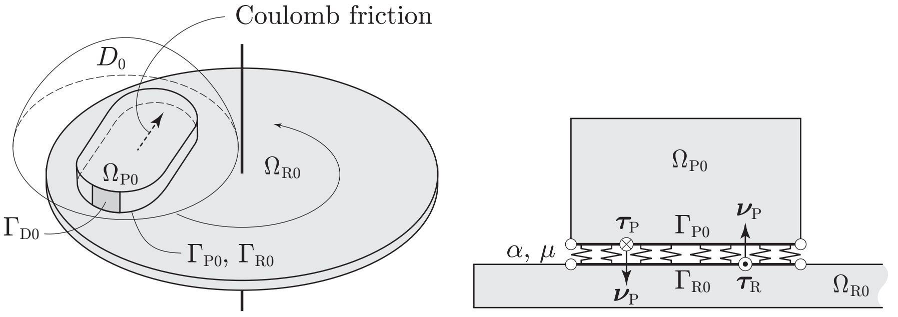
1. Abstract
The present paper describes a solution to a non-parametric shape optimization problem of a brake model suppressing squeal noise. The brake model consists of a rotor and a pad between which the Coulomb friction occurs. A main problem is defined as a complex eigenvalue problem of the brake model obtained from the equation of motion. As an objective cost function, we use the positive real part of the complex eigenvalue causing the brake squeal. The volume of the pad is used as a constraint cost function. The Fréchet derivative of the objective cost function with respect to the domain variation, which we call the shape derivative of the objective cost function, is evaluated using the solution of the main problem and the adjoint problem. A scheme to solve the shape optimization problem is presented using an iterative algorithm based on the H^1 gradient method (the traction method) for reshaping. A numerical result of a simple rotor-pad model illustrates that the real part of the target complex eigenvalue monotonously decreases satisfying the volume constraint.
2. Keywords: shape optimization, brake squeal, complex eigenvalue, self excited vibration, H^1 gradient method, traction method

3. Introduction
Brake squeal is known as a vibration phenomenon in the frequency range between 1 and 15 kHz caused by the friction between the rotor and the pad. Since it causes customer dissatisfaction, some effective method to prevent it in design stage is strongly desired.
Until now, many studies have been conducted in order to unravel brake squeal phenomenon. Mills [1] explained the brake squeal using the stick-slip vibration phenomenon caused by the friction force. North [2] introduced a simple model of a rotor and a pad between which the Coulomb friction occurs, and considered that the brake squeal is a self-excited vibration induced by the friction force. Based on North's idea, Millner [3] revealed that the stiffness matrix becomes asymmetric in the rotor and pad model with the Coulomb friction, and that the natural vibrations are determined by solutions of a complex eigenvalue problem. Then, he pointed out, if the real part of a complex eigenvalue is positive, a dynamic instability occurs. Many researchers analyzed the dynamic instability with the asymmetric stiffness matrix by the finite element method [4, 5].
Moreover, the researches finding the optimum shape which minimize the positive real part of the complex eigenvalue have been started since the 2000s. Lee et al. [6] and Guan et al. [7] presented formulations of parametric optimization problem by choosing eigenvalues of the components in the brake model as the design variables and the real part of the complex eigenvalue causing the brake squeal as the objective cost function, and showed numerical examples. Based on the assumption that the ideal eigenvalues reducing the positive real part of the complex eigenvalue were determined for the components in the brake model, Goto et al. [8] presented a method to find the shapes of the components as a solution of the non-parametric shape optimization problem using the error of the eigenvalues from the ideal values as the objective cost function.
In recent years, non-parametric optimization methods are applied to the optimum design problems in the brake model. Nelagadde et al. [9] presented a method to obtain optimum shapes of the components of the brake model to increase frequency separation between the critical modes while constraining the frequency separation between other selected modes by using a commercial software. Soh et al. [10] analyzed the optimum shape of the caliper housing by the topology optimization method using the real part of the complex eigenvalue as the objective function. However, an approach based on a formulation of the non-parametric shape optimization problem using the real part of the complex eigenvalue as the objective function has not been presented yet.
In the present paper, we formulate a shape optimization problem of a brake model consisting of a rotor and a pad between which the Coulomb friction occurs, and presents the solution of the problem.



(a) Initial domains Ω_{R0} and Ω_{P0} (b) Friction boundaries Γ_{R0} and Γ_{P0}

Figure 1: Brake model

We discuss these as follows. In Section 4, we define initial domains of the brake model and choose mapping from the initial domain of the pad to varied domain as design variable. Using the domains, in Section 5, we formulate the complex eigenvalue problem of the natural vibrations as a main problem in shape optimization problem. In Section 6, using the solution of the main problem, we formulate a shape optimization problem using the real part of an eigenvalue as an objective function and the volume of the pad as a constraint function. The evaluation methods for the shape derivatives of the cost functions are shown in Section 7. Using these shape derivatives of the cost functions, we present in Section 8 a method to obtain the domain mappings that decrease the cost functions. A scheme to solve the shape optimization problem with constraints is presented in Section 9. Finally, in Section 10, we show the numerical result for shape optimization of a simple brake model.

4. Brake model

Let us define initial domains for a brake model as depicted in Fig. 1. Let Ω_{R0} and Ω_{P0} be $d \in \{2,3\}$ dimensional bounded domains of linear elastic continua denoting a rotor and a pad, respectively. Γ_{R0} and Γ_{P0} denote contact boundaries on the boundary of rotor $\partial\Omega_{R0}$ and the boundary of pad $\partial\Omega_{P0}$, respectively. Let $\boldsymbol{\nu}_R$ and $\boldsymbol{\nu}_P$ be the normals, and $\boldsymbol{\tau}_R$ and $\boldsymbol{\tau}_P$ be the tangents on Γ_{R0} and Γ_{P0}, respectively. In the present paper, we assume that Ω_{P0} is variable. To define a shape optimization problem of Ω_{P0}, $\partial\Omega_{P0}$ is required to be at least the Lipschitz boundary, i.e. the $C^{0,1}$ class.

In the present paper, we use the notation $W^{s,p}\left(\Omega_0;\mathbb{R}^d\right)$ to denote the Sobolev space for the set of functions defined in Ω_0 and having values in $\mathbb{R}^d$ that are $s \in [0,\infty]$ times differentiable and $p \in [1,\infty]$-th order Lebesgue integrable, and call its smoothness the $W^{s,p}$ class. The notation $H^s\left(\Omega_0;\mathbb{R}^d\right)$ and $C^{s,\alpha}$ for $\alpha \in (0,1]$ are used as $W^{s,2}\left(\Omega_0;\mathbb{R}^d\right)$ and $W^{s+\alpha,\infty}\left(\Omega_0;\mathbb{R}^d\right)$.

Moreover, we assume that domain variation of Ω_{P0} as follows. Let D_0 be a fixed domain such that $D_0 \supset \Omega_{P0}$. Denoting $D_0 \cup \partial D_0$ by $\bar{D}_0$, domain variation of Ω_{P0} is given by a map $\boldsymbol{\phi} : \bar{D}_0 \to \mathbb{R}^d$ as shown in Fig. 2 belonging to the admissible set

$$\mathcal{D} = \Big\{ \boldsymbol{\phi} \in W^{1,\infty}\left(D_0;\ \mathbb{R}^d\right) \ \Big|\ \|\boldsymbol{\phi} - \boldsymbol{\phi}_0\|_{W^{1,\infty}(D_0;\ \mathbb{R}^d)} < 1, \quad \boldsymbol{\phi}\left(\Omega_{P0}\right) \subseteq D_0,\ \boldsymbol{\phi} = \boldsymbol{\phi}_0 \text{ on } \Gamma_{P0} \cup \Gamma_{D0} \Big\} \tag{1}$$

where, $\boldsymbol{\phi}_0$ is an identity mapping such as $\boldsymbol{\phi}_0\left(\boldsymbol{x}\right) = \boldsymbol{x}$ for $\boldsymbol{x} \in D_0$. $\|\boldsymbol{\phi} - \boldsymbol{\phi}_0\|_{W^{1,\infty}(D_0;\ \mathbb{R}^d)} < 1$ is used so that $\boldsymbol{\phi} \in \mathcal{D}$ is a one-to-one mapping. With respect to $\boldsymbol{\phi} \in \mathcal{D}$, we denote the new domain $\{\boldsymbol{\phi}\left(\boldsymbol{x}\right) \mid \boldsymbol{x} \in \Omega_{P0}\}$ as $\Omega_P\left(\boldsymbol{\phi}\right)$.

5. Main problem

Using the domains for the brake model, let us define a main problem for brake squeal. At first, let us consider the natural vibration of the brake model of Fig. 1.

Let $\boldsymbol{u}$ be the displacement expressing natural vibration and its admissible set is given for $q > d$ as

$$\mathcal{U} = \left\{ \boldsymbol{u} \in W^{2,2q}\left(D_0 \times \mathbb{R};\mathbb{R}^d\right) \mid \boldsymbol{u} = \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \Gamma_{P0} \cup \Gamma_{D0} \right\}. \tag{2}$$

The condition that $\boldsymbol{u}$ belongs to $W^{2,2q}$ class will be used in the process of deriving the shape derivative of the objective cost function after converting $\boldsymbol{u}$ into the eigenmode $\hat{\boldsymbol{u}}_k$ for $k \in \{1,2,\cdots\}$ belonging to $\mathcal{S}$ defined in Eq. (4).

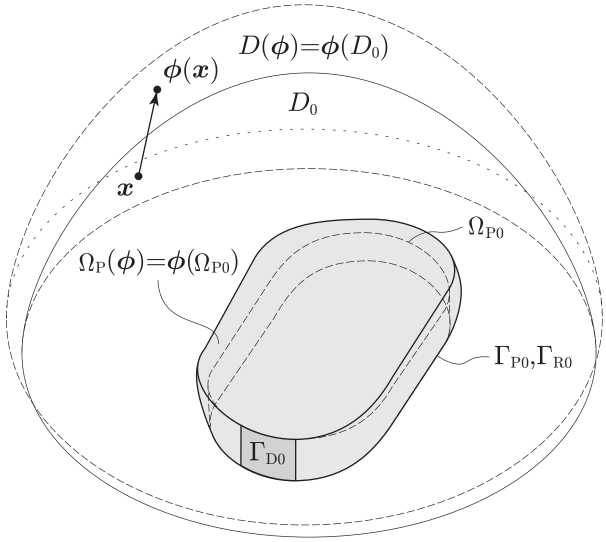


Figure 2: Domain variation of pad

In the present paper, let $\boldsymbol{u}_{\mathrm{R}}$ and $\boldsymbol{u}_{\mathrm{P}}$ denote the displacements $\boldsymbol{u}$ in $\Omega_{\mathrm{R}}(\boldsymbol{\phi})$ and $\Omega_{\mathrm{P}}(\boldsymbol{\phi})$, respectively. Let $\boldsymbol{E}(\boldsymbol{u}) = \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u}^{\mathrm{T}} + \left(\boldsymbol{\nabla}\boldsymbol{u}^{\mathrm{T}}\right)^{\mathrm{T}}\right)$ and $\boldsymbol{S}(\boldsymbol{u}) = \boldsymbol{C}\boldsymbol{E}(\boldsymbol{u})$ denote strain tensor and Cauchy stress tensor, respectively. Moreover, let α denote stiffness on relation between rotor and pad, μ denote the coefficient of the Coulomb friction, ρ_{R} and ρ_{P} denote the densities of the rotor and the pad, respectively. In this paper, we assume that α, μ, ρ_{R} and ρ_{P} are given as positive constants.
Based on these definitions and the notation $\dot{(\,\cdot\,)}$ for the time derivative, let us define the equations of motion for the brake model.

Problem 1 (Free vibration problem) For $\boldsymbol{\phi} \in \mathcal{D}$ and intial displacement $\bar{\boldsymbol{u}}_0 \in W^{2,2q}\left(D_0;\mathbb{R}^d\right)$ and initial velocity $\bar{\boldsymbol{v}}_0 \in W^{2,2q}\left(D_0;\mathbb{R}^d\right)$, find $\boldsymbol{u} \in \mathcal{S}$ such that

$$
\begin{aligned}
&\rho_{\mathrm{R}}\ddot{\boldsymbol{u}}_{\mathrm{R}} - \left(\boldsymbol{\nabla}\cdot\boldsymbol{S}\left(\boldsymbol{u}_{\mathrm{R}}\right)\right)^{\mathrm{T}} = \boldsymbol{0}_{\mathbb{R}^d} \quad \text{in } \Omega_{\mathrm{R}}(\boldsymbol{\phi}) \times \mathbb{R},\\
&\rho_{\mathrm{P}}\ddot{\boldsymbol{u}}_{\mathrm{P}} - \left(\boldsymbol{\nabla}\cdot\boldsymbol{S}(\boldsymbol{u}_{\mathrm{P}})\right)^{\mathrm{T}} = \boldsymbol{0}_{\mathbb{R}^d} \quad \text{in } \Omega_{\mathrm{P}}(\boldsymbol{\phi}) \times \mathbb{R},\\
&\boldsymbol{S}(\boldsymbol{u}_{\mathrm{R}})\boldsymbol{\nu}_{\mathrm{R}} = \boldsymbol{0}_{\mathbb{R}^d} \quad \text{on } \left(\partial\Omega_{\mathrm{R}}(\boldsymbol{\phi}) \setminus \Gamma_{\mathrm{R0}}\right) \times \mathbb{R},\\
&\boldsymbol{S}(\boldsymbol{u}_{\mathrm{P}})\boldsymbol{\nu}_{\mathrm{P}} = \boldsymbol{0}_{\mathbb{R}^d} \quad \text{on } \left(\partial\Omega_{\mathrm{P}}(\boldsymbol{\phi}) \setminus \Gamma_{\mathrm{P0}}\right) \times \mathbb{R},\\
&\boldsymbol{S}(\boldsymbol{u}_{\mathrm{R}})\boldsymbol{\nu}_{\mathrm{R}} = \alpha\left\{(\boldsymbol{u}_{\mathrm{R}} - \boldsymbol{u}_{\mathrm{P}})\cdot\boldsymbol{\nu}_{\mathrm{R}}\right\}\boldsymbol{\nu}_{\mathrm{R}} \quad \text{on } \Gamma_{\mathrm{R0}} \times \mathbb{R},\\
&\boldsymbol{S}(\boldsymbol{u}_{\mathrm{R}})\boldsymbol{\tau}_{\mathrm{R}} = \mu\alpha\left\{(\boldsymbol{u}_{\mathrm{R}} - \boldsymbol{u}_{\mathrm{P}})\cdot\boldsymbol{\nu}_{\mathrm{R}}\right\}\boldsymbol{\tau}_{\mathrm{R}} \quad \text{on } \Gamma_{\mathrm{R0}} \times \mathbb{R},\\
&\boldsymbol{S}(\boldsymbol{u}_{\mathrm{P}})\boldsymbol{\nu}_{\mathrm{P}} = \alpha\left\{(\boldsymbol{u}_{\mathrm{P}} - \boldsymbol{u}_{\mathrm{R}})\cdot\boldsymbol{\nu}_{\mathrm{P}}\right\}\boldsymbol{\nu}_{\mathrm{P}} \quad \text{on } \Gamma_{\mathrm{P0}} \times \mathbb{R},\\
&\boldsymbol{S}(\boldsymbol{u}_{\mathrm{P}})\boldsymbol{\tau}_{\mathrm{P}} = -\mu\alpha\left\{(\boldsymbol{u}_{\mathrm{P}} - \boldsymbol{u}_{\mathrm{R}})\cdot\boldsymbol{\nu}_{\mathrm{P}}\right\}\boldsymbol{\tau}_{\mathrm{P}} \quad \text{on } \Gamma_{\mathrm{P0}} \times \mathbb{R}\\
&\boldsymbol{u}_{\mathrm{R}} = \boldsymbol{u}_{\mathrm{P}} \quad \text{on } \left(\Gamma_{\mathrm{R0}} \cup \Gamma_{\mathrm{P0}}\right) \times \mathbb{R},\\
&\boldsymbol{u} = \bar{\boldsymbol{u}}_0 \quad \text{in } \Omega_{\mathrm{R}}(\boldsymbol{\phi}) \cup \Omega_{\mathrm{P}}(\boldsymbol{\phi}) \times \{0\},\\
&\dot{\boldsymbol{u}} = \bar{\boldsymbol{v}}_0 \quad \text{in } \Omega_{\mathrm{R}}(\boldsymbol{\phi}) \cup \Omega_{\mathrm{P}}(\boldsymbol{\phi}) \times \{0\},\\
&\boldsymbol{u} = \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \Gamma_{\mathrm{D0}} \times \mathbb{R}.
\end{aligned}
$$

Here, the negative sign of the Coulomb friction in the equation on $\Gamma_{\mathrm{P0}} \times \mathbb{R}$ makes the term of strain energy in the weak form of Problem 1 be asymmetric with respect to $\boldsymbol{u}$ and its adjoint function. Then, the eignvalue problem for the natural vibrations of the brake model becomes a complex eigenvalue problem. Since this brake model is a linear system with respect to $\boldsymbol{u}$, the form of separation of variables is given for some $s \in \mathbb{C}$ as

$$
\boldsymbol{u}(\boldsymbol{x},t) = \mathrm{e}^{st}\hat{\boldsymbol{u}}(\boldsymbol{x}) + \mathrm{e}^{s^{\mathrm{c}}t}\hat{\boldsymbol{u}}^{\mathrm{c}}(\boldsymbol{x}), \tag{3}
$$

where $(\,\cdot\,)^{\mathrm{c}}$ denotes the complex conjugation. From the definition of $\mathcal{U}$ in Eq. (2), the admissible set for $\hat{\boldsymbol{u}}$ is given by

$$
\mathcal{S} = \left\{ \hat{\boldsymbol{u}} \in W^{2,2q}\left(D_0;\mathbb{C}^d\right) \;\middle|\; \hat{\boldsymbol{u}} = \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \Gamma_{\mathrm{P0}} \cup \Gamma_{\mathrm{D0}} \right\}. \tag{4}
$$

By substituting Eq. (3) into Problem 1, we have a complex eigenvalue problem for natural vibrations. For compact expression of the weak form, we define the Lagrange function of the complex eigenvalue problem as

$$\mathscr{L}_{\mathrm{M}}\left(\boldsymbol{\phi}, s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}\right) = h\left(s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}^{\mathrm{c}}\right) + h\left(s_k^{\mathrm{c}}, \hat{\boldsymbol{u}}_k^{\mathrm{c}}, \hat{\boldsymbol{v}}\right) \tag{5}$$

for $(s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}) \in \mathbb{C} \times \mathcal{S} \times \mathcal{S}$ for $k \in \{1, 2, \cdots\}$, where

$$\begin{aligned} h\left(s, \hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}^{\mathrm{c}}\right) = {} & a_{\mathrm{R}}\left(\hat{\boldsymbol{u}}_{\mathrm{R}}, \hat{\boldsymbol{v}}_{\mathrm{R}}^{\mathrm{c}}\right) + s^2 b_{\mathrm{R}}\left(\hat{\boldsymbol{u}}_{\mathrm{R}}, \hat{\boldsymbol{v}}_{\mathrm{R}}^{\mathrm{c}}\right) - c_{\mathrm{R}}\left(\hat{\boldsymbol{u}}_{\mathrm{R}} - \hat{\boldsymbol{u}}_{\mathrm{P}}, \hat{\boldsymbol{v}}_{\mathrm{R}}^{\mathrm{c}}\right) - d_{\mathrm{R}}\left(\hat{\boldsymbol{u}}_{\mathrm{R}} - \hat{\boldsymbol{u}}_{\mathrm{P}}, \hat{\boldsymbol{v}}_{\mathrm{R}}^{\mathrm{c}}\right) \\ & + a_{\mathrm{P}}\left(\hat{\boldsymbol{u}}_{\mathrm{P}}, \hat{\boldsymbol{v}}_{\mathrm{P}}^{\mathrm{c}}\right) + s^2 b_{\mathrm{P}}\left(\hat{\boldsymbol{u}}_{\mathrm{P}}, \hat{\boldsymbol{v}}_{\mathrm{P}}^{\mathrm{c}}\right) - c_{\mathrm{P}}\left(\hat{\boldsymbol{u}}_{\mathrm{P}} - \hat{\boldsymbol{u}}_{\mathrm{R}}, \hat{\boldsymbol{v}}_{\mathrm{P}}^{\mathrm{c}}\right) + d_{\mathrm{P}}\left(\hat{\boldsymbol{u}}_{\mathrm{P}} - \hat{\boldsymbol{u}}_{\mathrm{R}}, \hat{\boldsymbol{v}}_{\mathrm{P}}^{\mathrm{c}}\right), \end{aligned} \tag{6}$$

and, for $(\,\cdot\,) \in \{\mathrm{P}, \mathrm{R}\}$,

$$\begin{aligned} a_{(\,\cdot\,)}\left(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}\right) &= \int_{\Omega_{(\,\cdot\,)}(\boldsymbol{\phi})} \boldsymbol{S}\left(\hat{\boldsymbol{u}}\right) \cdot \boldsymbol{E}\left(\hat{\boldsymbol{v}}\right) \,\mathrm{d}x, \\ b_{(\,\cdot\,)}\left(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}\right) &= \int_{\Omega_{(\,\cdot\,)}(\boldsymbol{\phi})} \rho_{(\,\cdot\,)} \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} \mathrm{d}x, \\ c_{(\,\cdot\,)}\left(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}\right) &= \int_{\Gamma_{(\,\cdot\,)0}} \alpha \left(\hat{\boldsymbol{v}} \cdot \boldsymbol{\nu}_{(\,\cdot\,)}\right) \left(\hat{\boldsymbol{v}} \cdot \boldsymbol{\nu}_{(\,\cdot\,)}\right) \mathrm{d}\gamma, \\ d_{(\,\cdot\,)}\left(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}\right) &= \int_{\Gamma_{(\,\cdot\,)0}} \mu\alpha \left(\hat{\boldsymbol{v}} \cdot \boldsymbol{\nu}_{(\,\cdot\,)}\right) \left(\hat{\boldsymbol{v}} \cdot \boldsymbol{\tau}_{(\,\cdot\,)}\right) \mathrm{d}\gamma. \end{aligned}$$

Using the definitions above, we define the weak form of the eigenvalue problem for natural vibrations of the brake model as follows.

Problem 2 (Eigenvalue problem for natural vibrations) For $\boldsymbol{\phi} \in \mathcal{D}$, find $(s_k, \hat{\boldsymbol{u}}_k) \in \mathbb{C} \times \mathcal{S}$ for $k \in \{1, 2, \cdots\}$ such that

$$\mathscr{L}_{\mathrm{M}}\left(\boldsymbol{\phi}, s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}\right) = 0$$

for all $\hat{\boldsymbol{v}} \in \mathcal{S}$.

6. Shape optimization problem

Using the solution s_k of Problem 2, let us define a shape optimization problem for the brake model. In the present paper, referring to the previous researches using the positive real part of the complex eigenvalue, we assume that the mode number k is given, and define an objective cost function as

$$f_0\left(\boldsymbol{\phi}, s_k\right) = 2\mathrm{Re}\left[s_k\right] = s_k + s_k^{\mathrm{c}}. \tag{7}$$

Moreover, we define a constraint cost function by the volume of the pad as

$$f_1\left(\boldsymbol{\phi}\right) = \int_{\Omega_{\mathrm{P}}(\boldsymbol{\phi})} \mathrm{d}x + c_1, \tag{8}$$

where c_1 is a positive constant for which there exists $\Omega_{\mathrm{P}}\left(\boldsymbol{\phi}\right)$ such that $f_1\left(\boldsymbol{\phi}\right) \leq 0$.
Useing these cost functions, we defined shape optimization problem as follow.

Problem 3 (Shape optimization problem) Let $\mathcal{D}$ and $\mathcal{S}$ be defined in Eq. (1) and Eq. (4). For $\boldsymbol{\phi} \in \mathcal{D}$, let $(s_k, \hat{\boldsymbol{u}}_k) \in \mathbb{C} \times \mathcal{S}$ be the solution of Problem 2 for given k. Let f_0 and f_1 are defined in Eq. (7) and Eq. (8), respectively. Find $\Omega_{\mathrm{P}}\left(\boldsymbol{\phi}\right)$ such that

$$\min_{\boldsymbol{\phi} \in \mathcal{D}} \left\{ f_0\left(\boldsymbol{\phi}, s_k\right) \mid f_1\left(\boldsymbol{\phi}\right) \leq 0,\ (s_k, \hat{\boldsymbol{u}}_k) \in \mathbb{C} \times \mathcal{S},\ \text{Problem 2} \right\}.$$

7. Shape derivative of cost functions

To solve Problem 3 by the gradient method, the Fréchet derivatives of the cost functions with respect to the domain variation, which we call the shape derivatives, are required. Then, let us derive the shape derivatives of f_0 and f_1 here.

Since the objective cost function $f_0(\boldsymbol{\phi}, s_k)$ contains s_k, we have to consider that the main problem is equality constraint. Hence, we put

$$\mathscr{L}_0(\boldsymbol{\phi}, s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0) = f_0(\boldsymbol{\phi}, s_k) - \mathscr{L}_{\mathrm{M}}(\boldsymbol{\phi}, s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}) \tag{9}$$

as the Lagrange function for f_0, where $\mathscr{L}_{\mathrm{M}}(\boldsymbol{\phi}, s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}})$ is defined in Eq. (5), and $\hat{\boldsymbol{v}}$ is used as the Lagrange multiplier for f_0. The shape derivative of $\mathscr{L}_0$ with respect to arbitrary domain variation $\boldsymbol{\varphi} \in W^{1,\infty}(D_0; \mathbb{R}^d)$ can be obtained, by applying the formulae of shape derivatives for domain and boundary integrals [11], as

$$\begin{aligned}\dot{\mathscr{L}}_0(\boldsymbol{\phi}, s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0)[\boldsymbol{\varphi}, s_k', \hat{\boldsymbol{u}}_k', \hat{\boldsymbol{v}}_0'] &= \mathscr{L}_{0\boldsymbol{\phi}}(\boldsymbol{\phi}, s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0)[\boldsymbol{\varphi}] + \mathscr{L}_{0s_k}(\boldsymbol{\phi}, s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0)[s_k'] \\ &\quad + \mathscr{L}_{0\hat{\boldsymbol{u}}_k}(\boldsymbol{\phi}, s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0)[\hat{\boldsymbol{u}}_k'] + \mathscr{L}_{0\hat{\boldsymbol{v}}_0}(\boldsymbol{\phi}, s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0)[\hat{\boldsymbol{v}}_0'],\end{aligned} \tag{10}$$

where $(s_k', \hat{\boldsymbol{u}}_k', \hat{\boldsymbol{v}}_0') \in \mathbb{C} \times \mathcal{S} \times \mathcal{S}$ denote the shape derivatives of $(s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0)$ with respect to domain variaiton $\boldsymbol{\varphi} \in W^{1,\infty}(D_0; \mathbb{R}^d)$.

The 4th term of the right-hand side of Eq. (10), which is written as

$$\begin{aligned}\mathscr{L}_{0\hat{\boldsymbol{v}}_0}(\boldsymbol{\phi}, s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0)[\hat{\boldsymbol{v}}_0'] &= -h_{\hat{\boldsymbol{v}}_0}(s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0^{\mathrm{c}})[\hat{\boldsymbol{v}}_0'] - h_{\hat{\boldsymbol{v}}_0}(s_k^{\mathrm{c}}, \hat{\boldsymbol{u}}_k^{\mathrm{c}}, \hat{\boldsymbol{v}}_0)[\hat{\boldsymbol{v}}_0'] \\ &= -h(s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0^{\mathrm{c}\prime}) - h(s_k^{\mathrm{c}}, \hat{\boldsymbol{u}}_k^{\mathrm{c}}, \hat{\boldsymbol{v}}_0'),\end{aligned} \tag{11}$$

becomes 0, if $(s_k, \hat{\boldsymbol{u}}_k)$ is the solution of Problem 2. On the other hand, the 2nd term of the right-hand side of Eq. (10) is written as

$$\begin{aligned}\mathscr{L}_{0s_k}(\boldsymbol{\phi}, s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0)[s_k'] &= f_{0s_k}(\boldsymbol{\phi}, s_k)[s_k'] - h_{s_k}(s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0^{\mathrm{c}})[s_k'] - h_{s_k}(s_k^{\mathrm{c}}, \hat{\boldsymbol{u}}_k^{\mathrm{c}}, \hat{\boldsymbol{v}}_0)[s_k'] \\ &= s_k' + s_k^{\mathrm{c}\prime} - 2s_k s_k' b_{\mathrm{R}}(\hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0^{\mathrm{c}}) - 2s_k^{\mathrm{c}} s_k^{\mathrm{c}\prime} b_{\mathrm{P}}(\hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0^{\mathrm{c}}) \\ &= s_k'(1 - 2s_k b_{\mathrm{R}}(\hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0^{\mathrm{c}})) + s_k^{\mathrm{c}\prime}(1 - 2s_k^{\mathrm{c}} b_{\mathrm{P}}(\hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0^{\mathrm{c}})).\end{aligned} \tag{12}$$

Moreover, the 3rd term of the right-hand side of Eq. (10) is written as

$$\begin{aligned}\mathscr{L}_{0\hat{\boldsymbol{u}}_k}(\boldsymbol{\phi}, s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0)[\hat{\boldsymbol{u}}_k'] &= -h_{\hat{\boldsymbol{u}}_k}(s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0^{\mathrm{c}})[\hat{\boldsymbol{u}}_k'] - h_{\hat{\boldsymbol{u}}_k}(s_k^{\mathrm{c}}, \hat{\boldsymbol{u}}_k^{\mathrm{c}}, \hat{\boldsymbol{v}}_0)[\hat{\boldsymbol{u}}_k'] \\ &= -h(s_k, \hat{\boldsymbol{u}}_k', \hat{\boldsymbol{v}}_0^{\mathrm{c}}) - h(s_k^{\mathrm{c}}, \hat{\boldsymbol{u}}_k^{\mathrm{c}\prime}, \hat{\boldsymbol{v}}_0).\end{aligned} \tag{13}$$

Then, Eq. (12) and Eq. (13) become 0, respectively, if $\hat{\boldsymbol{v}}_0$ is the solution of the following weak form of the adjoint problem.

Problem 4 (Adjoint problem for f_0) For $\boldsymbol{\phi} \in \mathcal{D}$, let $(s_k, \hat{\boldsymbol{u}}_k)$ be the solution of Problem 2 for k. Find $\hat{\boldsymbol{v}}_0 \in \mathcal{S}$ such that

$$h(s_k, \hat{\boldsymbol{u}}_k', \hat{\boldsymbol{v}}_0^{\mathrm{c}}) + h(s_k^{\mathrm{c}}, \hat{\boldsymbol{u}}_k^{\mathrm{c}\prime}, \hat{\boldsymbol{v}}_0) = 0 \tag{14}$$

for all $\hat{\boldsymbol{u}}_k' \in \mathcal{S}$, and

$$2s_k b(\hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0^{\mathrm{c}}) = 2s_k^{\mathrm{c}} b(\hat{\boldsymbol{u}}_k^{\mathrm{c}}, \hat{\boldsymbol{v}}_0) = 1. \tag{15}$$

For the solution $\hat{\boldsymbol{v}}_0$ of Problem 4, from Eq. (14), we have

$$\hat{\boldsymbol{v}}_0 = c\hat{\boldsymbol{u}}_k$$

for all $c \in \mathbb{C}$. Moreover, by using Eq. (15), we have

$$c = \frac{1}{2s_k b(\hat{\boldsymbol{u}}_k, \hat{\boldsymbol{u}}_k^{\mathrm{c}})}.$$

Then, $\hat{\boldsymbol{v}}_0$ is obtained by normalization of $\hat{\boldsymbol{u}}_k$ with c above.

Based on the results, if s_k, $\hat{\boldsymbol{u}}_k$ and $\hat{\boldsymbol{v}}_0$ are the solutions of Problem 2 and Problem 4, respectively, $\dot{\mathscr{L}}_0(\boldsymbol{\phi}, s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0)[\boldsymbol{\varphi}, s'_k, \hat{\boldsymbol{u}}'_k, \hat{\boldsymbol{v}}'_0]$ in Eq. (10) becomes

$$\mathscr{L}_{0\boldsymbol{\phi}}(\boldsymbol{\phi}, s_k, \hat{\boldsymbol{u}}_k, \hat{\boldsymbol{v}}_0)[\boldsymbol{\varphi}] = \int_{\partial\Omega_{\mathrm{P}}(\boldsymbol{\phi})\setminus(\Gamma_{\mathrm{P}0}\cup\Gamma_{\mathrm{D}0})} \boldsymbol{g}_{\partial\Omega 0} \cdot \boldsymbol{\varphi}\,\mathrm{d}\gamma = \langle \boldsymbol{g}_0, \boldsymbol{\varphi} \rangle, \tag{16}$$

where

$$\boldsymbol{g}_{\partial\Omega 0} = 2\mathrm{Re}\left[\boldsymbol{S}(\hat{\boldsymbol{u}}_k)\cdot\boldsymbol{E}(\hat{\boldsymbol{v}}_0^{\mathrm{c}}) + s_k^2\rho_{\mathrm{P}}\hat{\boldsymbol{u}}_k\cdot\hat{\boldsymbol{v}}_0^{\mathrm{c}}\right]\boldsymbol{\nu}_{\mathrm{P}}. \tag{17}$$

Here, we used the condition $\boldsymbol{\varphi} = \boldsymbol{0}_{\mathbb{R}^d}$ on $\Gamma_{\mathrm{P}0}\cup\Gamma_{\mathrm{D}0}$ as given in Eq. (1).
On the other hand, for $f_1(\boldsymbol{\phi})$, we have

$$f'_1(\boldsymbol{\phi})[\boldsymbol{\varphi}] = \int_{\partial\Omega_{\mathrm{P}}(\boldsymbol{\phi})\setminus(\Gamma_{\mathrm{P}1}\cup\Gamma_{\mathrm{D}0})} \boldsymbol{g}_{\partial\Omega 0} \cdot \boldsymbol{\varphi}\,\mathrm{d}\gamma = \langle \boldsymbol{g}_1, \boldsymbol{\varphi} \rangle. \tag{18}$$

where

$$\boldsymbol{g}_{\partial\Omega 1} = \boldsymbol{\nu}_{\mathrm{P}}. \tag{19}$$

We call $\boldsymbol{g}_0$ and $\boldsymbol{g}_1$ the shape derivatives of f_0 and f_1, respectively.

8. The H^1 gradient method

The H^1 gradient method is proposed as a method for finding the variation of the design variable, such as the domain mapping or the density parameter that decreases a cost function, as a solution to a boundary value problem of an elliptic partial differential equation [12–14]. In the case that a shape derivative $\boldsymbol{g}_i$ of a cost function $f_i(\boldsymbol{\phi})$ for $i \in \{0, 1\}$, the H^1 gradient method can be described as follows.

Problem 5 (H^1 gradient method for shape optimization) Let X be a Hilbert space of $H^1(D_0; \mathbb{R}^d)$, and let $a: X \times X \to \mathbb{R}$ be a coercive bilinear form on X such that there exists $\beta > 0$ that satisfies

$$a(\boldsymbol{w}, \boldsymbol{w}) \geq \beta \|\boldsymbol{w}\|_X^2$$

for all $\boldsymbol{w} \in X$. For $\boldsymbol{g}_i \in X'$ (dual space of X), which is a Fréchet derivative of cost function $f(\boldsymbol{\phi})$ at $\boldsymbol{\phi} \in X$, find $\boldsymbol{\varphi}_{gi} \in X$ such that

$$a(\boldsymbol{\varphi}_{gi}, \boldsymbol{w}) = -\langle \boldsymbol{g}_i, \boldsymbol{w} \rangle \tag{20}$$

for all $\boldsymbol{w} \in X$.

Problem 5 can be solved numerically with the standard finite element method by considering that Eq. (20) is a weak form of a boundary value problem of an elliptic partial differential equation. In the present paper, we use

$$a(\boldsymbol{\varphi}, \boldsymbol{\psi}) = c_a \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{S}(\boldsymbol{\varphi}) \cdot \boldsymbol{E}(\boldsymbol{\psi})\,\mathrm{d}x \tag{21}$$

for $\boldsymbol{\varphi} \in \mathcal{S}$ and $\boldsymbol{\psi} \in \mathcal{S}$, where $\boldsymbol{E}(\,\cdot\,)$ and $\boldsymbol{S}(\,\cdot\,)$ are the same as in Problem 2, and c_a is a positive constant. The coerciveness is secured by the Dirichlet condition on $\Gamma_{\mathrm{P}0}\cup\Gamma_{\mathrm{D}0}$ in Eq. (4). The strong form of the H^1 gradient method using Eq. (21) is written as follows.

Problem 6 (H^1 gradient method for Problem 3) For $\boldsymbol{g}_i$, find $\boldsymbol{\varphi}_{gi} \in H^1(\Omega(\boldsymbol{\phi}); \mathbb{R}^d)$ such that

$$\begin{aligned}
&-c_a\boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{S}(\boldsymbol{\varphi}_{gi}) = \boldsymbol{0}_{\mathbb{R}^d}^{\mathrm{T}} \quad \text{in } \Omega_{\mathrm{P}}(\boldsymbol{\phi}),\\
&c_a\boldsymbol{S}(\boldsymbol{\varphi}_{gi})\boldsymbol{\nu} = -\boldsymbol{g}_{\partial\Omega i} \quad \text{on } \partial\Omega_{\mathrm{P}}(\boldsymbol{\phi}) \setminus \left(\bar{\Gamma}_{\mathrm{P}0}\cup\bar{\Gamma}_{\mathrm{D}0}\right),\\
&\boldsymbol{\varphi}_{gi} = \boldsymbol{0}_{\mathbb{R}^d} \quad \text{on } \Gamma_{\mathrm{P}0}\cup\Gamma_{\mathrm{D}0}.
\end{aligned}$$

Figure 3 shows the boundary condition of Problem 6.
If $\hat{\boldsymbol{u}}$ satisfies the conditions in $\mathcal{S}$, we can confirm that the solution $\boldsymbol{\varphi}_{gi}$ of Problem 5 belongs to $W^{1,\infty}(D_0; \mathbb{R}^d)$.

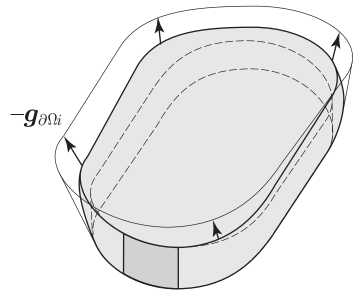


Figure 3: H^1 gradient method using Dirichlet condition

9. Solution to the shape optimization problem

To solve Problem 3, we use an iterative method based on sequential quadratic programming. The domain variation decreasing $f_0(\boldsymbol{\phi}, s_k)$ while satisfying $f_1(\boldsymbol{\phi}) \le 0$ is determined with the solution of the following problem. In this section, we denote $f_0(\boldsymbol{\phi}, s_k)$ as $f_0(\boldsymbol{\phi})$ and its shape derivative as $\boldsymbol{g}_0$.

Problem 7 (SQ approximation) For $\boldsymbol{\phi} \in \mathcal{D}$, let $\boldsymbol{g}_i$ be the shape derivatives of $f_i(\boldsymbol{\phi})$ for $i \in \{0, 1\}$, and let $f_1(\boldsymbol{\phi}) \le 0$. Let $a(\,\cdot\,,\,\cdot\,)$ be given as in Eq. (20). Find $\boldsymbol{\varphi}$ such that

$$\min_{\boldsymbol{\varphi} \in W^{1,\infty}(D_0;\mathbb{R}^d)} \left\{ q(\boldsymbol{\varphi}) = \frac{1}{2} a(\boldsymbol{\varphi}, \boldsymbol{\varphi}) + \langle \boldsymbol{g}_0, \boldsymbol{\varphi} \rangle \;\middle|\; f_1(\boldsymbol{\phi}) + \langle \boldsymbol{g}_1, \boldsymbol{\varphi} \rangle \le 0 \right\}.$$

The Lagrange function of Problem 8 is defined as

$$\mathscr{L}_{\mathrm{SQ}}(\boldsymbol{\varphi}, \lambda_1) = q(\boldsymbol{\varphi}) + \lambda_1 (f_1(\boldsymbol{\phi}) + \langle \boldsymbol{g}_1, \boldsymbol{\varphi} \rangle)$$

where $\lambda_1 \in \mathbb{R}$ is the Lagrange multiplier for the constraint $f_1(\boldsymbol{\varphi}) \le 0$. The Karush–Kuhn–Tucker conditions for Problem 8 are given as

$$a(\boldsymbol{\varphi}, \boldsymbol{\varphi}) + \langle \boldsymbol{g}_0 + \lambda_1 \boldsymbol{g}_1, \boldsymbol{\varphi} \rangle = 0, \tag{22}$$

$$f_1(\boldsymbol{\phi}) + \langle \boldsymbol{g}_1, \boldsymbol{\varphi} \rangle \le 0, \tag{23}$$

$$\lambda_1 (f_1(\boldsymbol{\phi}) + \langle \boldsymbol{g}_1, \boldsymbol{\varphi} \rangle) = 0, \tag{24}$$

$$\lambda_1 \ge 0 \tag{25}$$

for all $\boldsymbol{\varphi} \in W^{1,\infty}(D_0; \mathbb{R}^d)$. Here, let $\boldsymbol{\varphi}_{gi}$ for $i \in \{0, 1\}$ be the solutions to Problem 5, and set

$$\boldsymbol{\varphi}_g = \boldsymbol{\varphi}_{g0} + \lambda_1 \boldsymbol{\varphi}_{g1}. \tag{26}$$

Then, by substituting $\boldsymbol{\varphi}_g$ of Eq. (26) for $\boldsymbol{\varphi}$ in Eq. (22), Eq. (22) holds. If the constraint in Eq. (23) is active, i.e. Eq. (23) holds with the equality, we have

$$\langle \boldsymbol{g}_1, \boldsymbol{\varphi}_{g1} \rangle \lambda_1 = -f_1(\boldsymbol{\phi}) + \langle \boldsymbol{g}_1, \boldsymbol{\varphi}_{g0} \rangle. \tag{27}$$

Equation (27) has a unique solution of λ_1. Moreover, if $f_1(\boldsymbol{\phi}) = 0$, we have

$$\langle \boldsymbol{g}_1, \boldsymbol{\varphi}_{g1} \rangle \lambda_1 = -\langle \boldsymbol{g}_1, \boldsymbol{\varphi}_{g0} \rangle. \tag{28}$$

Since Eq. (28) is independent of the magnitude of $\boldsymbol{\varphi}_{g0}$ and $\boldsymbol{\varphi}_{g1}$ to determine λ_1, Eq. (28) is used in the numerical scheme for the initial domain Ω_0 in which we assume $f_1(\boldsymbol{\phi}) = 0$ is satisfied. If $\lambda_1 < 0$ in the solution λ_1 to Eq. (27) or Eq. (28), by putting $\lambda_1 = 0$, we have λ_1 satisfying Eq. (22) to Eq. (25). The detail of the numerical scheme is shown in the previous paper [14].
The magnitude of $\boldsymbol{\varphi}_g$ in Eq. (26), which means the step size for domain variation, is adjusted by selection of c_a in Eq. (21) using criteria such as the Armijo and Wolfe's criteria to ensure the global convergence in Problem 7. The outline of the numerical scheme is shown in Fig. 4. The detail is shown in the previous paper [14].

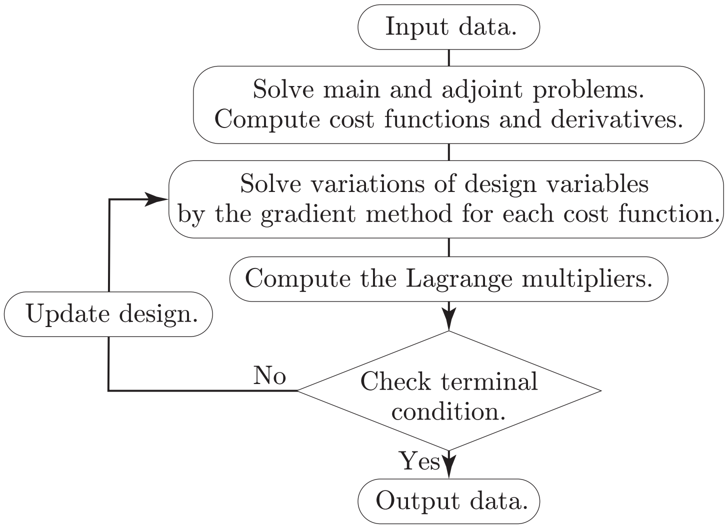


Figure 4: Sequential quadratic approximation method

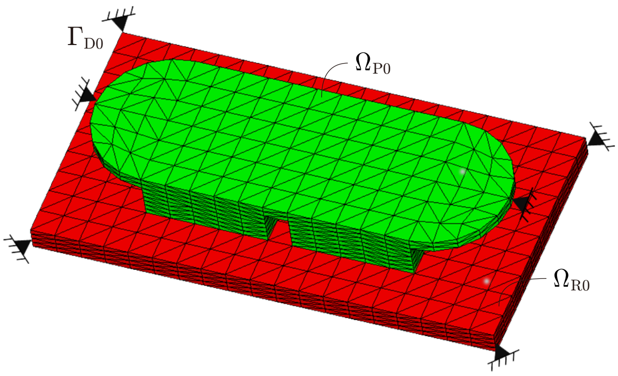


Figure 5: Finite element model

Table 1: Numerical results of complex eigenvalues

k	Re	Im
1	-1.189E+04	1.2836E+05
2	-1.922E+04	1.8024E+05
3	-3.093E+04	3.2078E+05
4	-3.331E+04	3.5971E+05
5	-3.732E+04	3.8596E+05
6	-5.122E+04	4.5752E+05
7	3.320E+04	5.0126E+05
8	-1.356E+05	5.2225E+05
9	-4.934E+04	5.3131E+05
10	-5.338E+04	5.3989E+05

(a) Initial shape

k	Re	Im
1	-1.199E+04	1.2834E+05
2	-1.916E+04	1.7289E+05
3	-3.056E+04	3.1931E+05
4	-3.471E+04	3.2881E+05
5	-3.702E+04	3.8540E+05
6	-4.936E+04	4.3695E+05
7	-2.158E+04	5.0336E+05
8	-5.636E+04	5.1569E+05
9	-5.548E+04	5.3397E+05
10	-1.604E+04	5.7519E+05

(b) Optimized shape

10. Numerical example

We developed an original computer program based on the numerical scheme described above using the finite element method to solve Problem 2 and Problem 4.

To show that the present method is effective, we solved a shape optimization problem of a simple brake model which finite element model is shown in Fig. 5. In this figure, the nodal points with the fixed signs

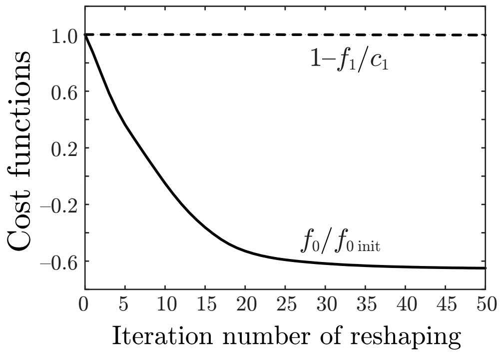
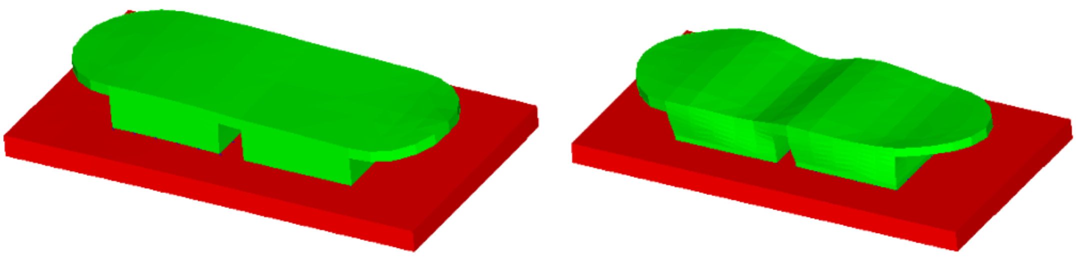


Figure 6: Iteration histories of cost functions with respect to reshaping



(a) Intitial shape

(b) Optimized shape

Figure 7: Comparison of shapes

are assumed to be fixed in Problem 2 and Problem 4, i.e. the Dirichlet condition is assigned. The length of the larger edge in $\Omega_{\mathrm{R}0}$ is 0.0095 [m]. We used 210 [GPa], 0.3 and 7.8×10^3 [kg/m^3] for Young's modulus, Poisson's ratio and the density of the rotor, respectively, and 16 [GPa], 0.3 and 2.1×10^3 [kg/m^3] for those of the pad. Moreover, 5.0×10^6 [N/m] and 0.05 are used for the contact stiffness α and the friction coefficient μ, respectively.

The numerical results of the complex eigenvalues for the initial shape, i.e. the numerical solution of Problem 2, is shown in Table 1 (a). Among the results, the 7th eigenvalue has a positive number in the real part. Then, we assumed $k = 7$ in Problem 3.

The iteration histories of cost functions f_0 and f_1 with respect to the number of reshaping are shown in Fig. 6. In this figure, $f_{0\,\mathrm{init}}$ and c_1 denote the value of f_0 and the volume of $\Omega_{\mathrm{P}0}$ at the initial shape, respectively. We see that f_0 decreases monotonically under satisfying the domain measure constraint of f_1. Table 1 (b) shows the numerical results of the complex eigenvalues after 50 times iteration of reshaping. We see that the 7th eigenvalue has a negative value in the real part. Figure 7 shows the comparison of the initial and optimized shapes. Remarkable change of the shape is observed.

11. Conclusions

In the present paper, we presented a numerical solution to shape optimization problems of a brake model consisting of a rotor and a pad between which the Coulomb friction occurs. The main problem was constructed as a complex eigenvalue problem of the brake model obtained from the equation of motion. The shape optimization problem was formulated with the positive real part of the complex eigenvalue assigned as a cause of brake squeal as an objective cost function, and the volume of the pad as a constraint cost function. The evaluation method of the shape derivative of the objective cost function was derived using the stationary conditions of the Lagrange function, and shown with the adjoint problem. A standard scheme to solve the shape optimization problem using the H^1 gradient method for reshaping was used to construct the numerical scheme. Finally, a numerical result for a simple rotor-pad model was illustrated. In the result, the real part of the target complex eigenvalue monotonously decreases satisfying the constraint for the volume of the pad. From the result, in the present paper, a methodology to find the optimum shape which minimize the real part of the complex eigenvalue assigned as a cause of brake squeal was presented.

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