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Robust topology optimization of 2D and 3D continuum and truss structures using a spectral stochastic finite element method

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1. Abstract
In this paper a framework is introduced for robust structural topology optimization for 2D and 3D continuum and truss problems. Spectral stochastic finite element method is used, with a polynomial chaos expansion to propagate uncertainties on the material characteristics to the response quantities. The uncertain parameters are modelled using a spatially correlated random field which is discretized using the Karhunen-Loève expansion. Special attention is paid to addressing the representation of the material uncertainties in linear truss elements. Several examples demonstrate the method on both 2D and 3D continuum and truss structures.
2. Keywords: Robust optimization, Truss optimization, Stochastic finite element method

3. Introduction
This research focusses on a novel robust structural topology optimization method for 2D and 3D continuum and truss problems. Structural optimization taking uncertainties into account is of significant importance to designers, since real-world structures require both efficient use of material and accurate modelling of material properties, manufacturing tolerances and loading of structures. When considering candidate designs, engineers are concerned with the sensitivity of the designs to small variations which can be quantified as uncertainties. Robust design optimization offers a framework for taking these uncertainties into account.
Uncertainties play an important role in engineering practice and are often accounted for using coefficients such as safety factors. While these coefficients provide a reliable margin, they are generally overly conservative and do not meet the needs of optimization procedures, which are also interested in the sensitivity of optimal solutions to perturbation, the *robustness* of the solution. The limited number of approaches to take these uncertainties into account in structural optimization are summarized in overviews by Tsompanakis et al. [1] and Schueller and Jensen [2]. The majority of the structural optimization studies accounting for uncertainty are concerned with shape optimization, while only a few studies deal with uncertainties in topology optimization. The limited number of works on the subject have been completed in the last decade or so, mostly focussing on random loading. Seepersad et al. [3] focus on designing mesoscopic material topology, where imperfections due to the manufacturing process are of great importance. An interval method is used to represent the uncertainties and a discrete ground structure approach to topology optimization of the material meso-structure is employed. Kogiso et al. [4] used the homogenization approach as a basis for a sensitivity-based RTO for compliant mechanisms, however, only random variation on the loading direction are considered. Conti et al. [5] formulated a Level Set (LS) based shape optimization method under stochastic loading, making use of a two-stage stochastic programming approach. De Gournay et al. [6] also used a LS approach to shape and topology optimization for minimal compliance, minimizing the 'worst case' compliance under perturbation of the loading. Guest and Igusa [7] used a mean compliance formulation under uncertainties on the nodal locations. Lógó et al. [8] developed a new loading criterion for compliance minimization for probabilistic loading, and extended this to uncertainties on the loading location [9]. Chen et al. [10] proposed a LS based robust shape and topology optimization (RSTO) method, taking material uncertainties into account. Asadpoure et al. [11] developed a robust formulation for mass minimization with uncertainties on the material properties, using a polynomial chaos approach concurrently with the development of the method presented in this

paper. Wang et al. [12] demonstrated a method for robust topology optimization applied to photonic waveguides using SIMP, with manufacturing uncertainties, by approximation by the threshold projection method. While some work has very recently been done on robust shape and topology optimization of two dimensional structures [13] for mass minimization, 3D structures appear to have not been dealt with broadly [10], and, to the authors' knowledge, no common frameworks exist for robust optimization of both continuum and truss structures. The representation of uncertainties in robust optimization of truss structures has also been relatively neglected and investigations thus far have failed to take some key features of trusses, such as element length, into account. All of these methods use an adaptation of a deterministic optimization algorithm to incorporate uncertainties. Starting from these considerations the remainder of the paper is ordered as follows: modelling of uncertainties is introduced in section 4, with the adaptation specifically for topology optimization considered in section 5. Computational examples of this method follow (section 6) and a discussion and suggestions for further work is then given (section 7).

4. Modelling of uncertainties for continuous and discrete structures

In this investigation the material uncertainties are expressed in terms of a spatially varying random field, which is discretized using a Karhunen-Loève (KL) expansion. Random fields allow for expression of spatially correlated random quantities, while being general enough to model uncorrelated quantities too. Spectral Stochastic Finite Element Method (SSFEM) [14] is used to derive the statistical measures of the response, allowing for a quantification of the terms of the objective function (a linear combination of the mean and standard deviation of the compliance), for a given volume fraction. Material models are generally expressed in terms of Gaussian or lognormal probability distributions, both of which can be taken into account in the SSFEM framework. In continuum structures the random field may be correlated over the entire domain, while in truss structures this is not the case. A novel analysis method for modelling the variation of material properties along the length of individual truss elements is developed, based on the SSFEM framework, and used for topology optimization of truss structures. Derivation of the objective function and the sensitivities necessary for the optimization procedure are demonstrated, making use of the response quantities. SSFEM discretization consists of series expansion of realizations of the original random field $H(\mathbf{x}, \theta)$ over a complete set of deterministic functions [14], where θ is a vector of random variables. The obtained series in then truncated after finite number of terms. Various discretization methods are available of which the Karhunen-Loève expansion (KL) is the most efficient in terms of the number of random variables required for a given accuracy [14], making it a good candidate for the computationally expensive task of design optimization. A Gaussian random field $H(\mathbf{x}, \theta)$ can be expanded as follows:

$$H(\mathbf{x}, \theta) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \xi_i(\theta) \varphi_i(\mathbf{x}) \tag{1}$$

where $\mu(\mathbf{x})$ is the mean value of the random phenomenon, λ_i's and φ_i's respectively the eigenvalues and eigenfunctions of the covariance kernel, and ξ_i's the random variables. The approximated field $\hat{H}$ can be found by truncating terms above some value M:

$$\hat{H} = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\lambda_i} \xi_i(\theta) \varphi_i(\mathbf{x}) \tag{2}$$

Truss analysis accounting for material property uncertainty is often achieved by associating a random variable with a the cross section area of each bar element [15, 11]. This approach has two fundamental shortcomings:

1. The approach presupposes a small scale for the problem, while trusses and individual truss elements are typically large in scale, and
2. the relative lengths of the elements are neglected in the probabilistic model.

At the scale of truss elements, often several meters in length, the variability of material properties along the length of the element can be very significant, spatially correlated quantities. Global 2D and 3D correlated random fields are not appropriate for modelling this variability, since no correlation exists between the material properties of separate elements. The proposed approach constructs individual 1D random fields across the individual truss elements, subdividing elements into segments. The compliance analysis and topology variables apply to the truss scale elements and nodes (figure 1). If each element is

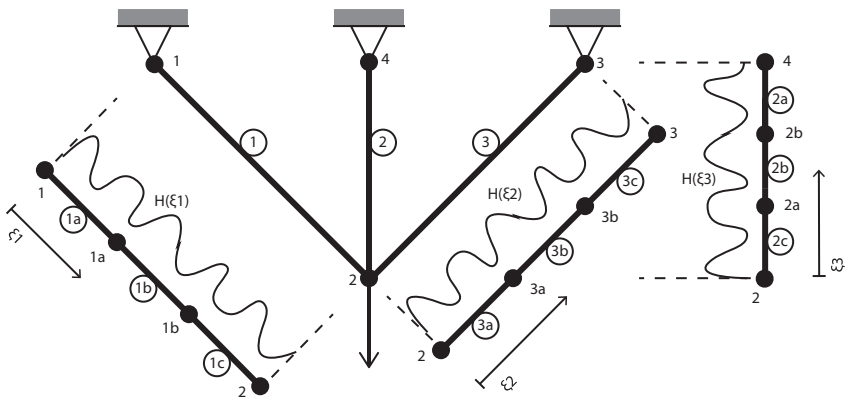


Figure 1: Truss element-level 1D random fields

substructured as shown in figure 1, a simple expression an be found to approximate the relative stiffness of the element as a whole, based on sampling the element-level field:

$$\hat{H}_e = \frac{1}{\sum\limits_{j=1}^{N_{se}} \frac{1}{\left(\mu_j(\mathbf{x}_j) + \sum\limits_{i=1}^{M} \sqrt{\lambda_i}\xi_i(\theta)\varphi_i(\mathbf{x}_j)\right)}} \tag{3}$$

where $\hat{H}_e$ is the element-level random field, μ_j is the mean value of the random field for sub-element j, and N_{se} is the total number of sub-elements. Since $\mu_j(\mathbf{x}_j)$ is constant for the element e, the following expression results:

$$\hat{H}_e = \mu(\mathbf{x}) + \sum_{i=1}^{M} \frac{\sqrt{\lambda_i}\xi_i(\theta)}{\sum\limits_{j=1}^{N_{se}} \frac{1}{\varphi_i(\mathbf{x}_j)}} \tag{4}$$

The remainder of the method is analogous to the continuum case.

5. Introducing uncertainties for robust topology optimization

5.1 Deterministic continuum topology optimization with the SIMP method

An important aspect of robust optimization consists in the modelling of the uncertainties to be included in the analysis portion of the optimization process. For this purpose a good understanding of the deterministic method (in this case SIMP) to be adapted to account for uncertainties is imperative. A classical way to state the (single objective) optimization problem is the minimization of some function f (the objective function) of the design variables $\mathbf{x}$, subject to some constraints g and h:

$$\begin{aligned} &\min_{\mathbf{x}} f(\mathbf{x}) \\ &\text{subject to: } \begin{cases} g(\mathbf{x}) \leq 0 \\ h(\mathbf{x}) = 0 \end{cases} \end{aligned} \tag{5}$$

In the case of compliance minimization the objective function can be written as follows:

$$f(\mathbf{x}) = C = \mathbf{f}^\top \mathbf{u} \tag{6}$$

where $\mathbf{f}$ is the external loading on the structure and $\mathbf{u}$ the nodal displacements. Typically the volume of the structure is constrained:

$$\frac{V(\mathbf{x})}{V_0} - c = 0 \tag{7}$$

where $V(\mathbf{x})$ and V_0 are respectively the volume of a design and the reference volume (the fraction of these quantities is called the volume fraction), and c is some constant chosen by the designer. The design

variables $\mathbf{x} = [x_1\ x_2\ \dots\ x_e \dots x_N]$ are scalers associated with element e where $e = 1 \dots N$ and N is the number of elements in the finite element mesh. For continuum problems x_e is a coefficient of the density of the element such that $x_{e,min} \leq x_e \leq 1$. SIMP has been exceedingly successful and implemented in numerous papers [16, 17, 18]. Using the SIMP formulation, the element stiffness matrix can be written as $\mathbf{K}_e = {x_e}^p \mathbf{K}_e^*$, where $\mathbf{K}_e^*$ is the stiffness matrix with density equal to the standard material density, and p is a penalty value chosen by the user (often taken as 3 for continuum structures) [18]. When truss structures are considered this type of penalization can be neglected by setting $p = 1$. The SIMP compliance objective function (6) and sensitivities are then calculated as follows:

$$C(\mathbf{x}) = \sum_{e=1}^{N} (x_e)^p \mathbf{u}_e^\top \mathbf{K}_e \mathbf{u}_e. \tag{8}$$

$$\frac{\partial C}{\partial x_e} = -p(x_e)^{p-1} \mathbf{u}_e^\top \mathbf{K}_e \mathbf{u}_e \tag{9}$$

Restrictions on the design space for continuum structures are essential for dealing with questions of existence of solutions [18]. Sigmund [19] introduced a mesh independency filtering technique which modifies the element sensitivities. Another method for ensuring existence of solutions was introduced by Guest et al. [20] using a minimum length scale.

5.2 Principle of the spectral stochastic finite element method

In the SIMP approach the element stiffness matrix $\mathbf{K}_e$ can then be written as:

$$\mathbf{K}_e(\theta) = \mathbf{K}_{e,0} + \sum_{i=1}^{M} \xi_i(\theta) \sqrt{\lambda_i} \int_{\Omega_e} \varphi_i(\mathbf{x}) \mathbf{B}^\top \mathbf{D}_0 \mathbf{B} d\Omega_e \tag{10}$$

where $\mathbf{K}_{e,0}$ is the deterministic element stiffness matrix, $\mathbf{B}$ is the matrix that relates the components of strain to the element nodal displacements, and $\mathbf{D}_0$ the deterministic elasticity matrix. Assembling matrices $\mathbf{K}_{e,i} = \sqrt{\lambda_i} \int_{\Omega_e} \varphi_i(\mathbf{x}) \mathbf{B}^\top \mathbf{D}_0 \mathbf{B} d\Omega_e$ to their global form $\mathbf{K}_i$ over the structural domain Ω_e, the equilibrium equation becomes:

$$\left(\mathbf{K}_0 + \sum_{i=1}^{M} \mathbf{K}_i \xi_i(\theta) \right) \mathbf{u}(\theta) = \mathbf{f} \tag{11}$$

Modelling of the response to a random process requires an expansion in which the covariance function need not be explicitly known [21]. The PCE assumes the random displacements $u(\theta)$ can be expanded as follows:

$$u(\theta) = \sum_{j=0}^{P-1} u_j \Psi_j(\theta) \tag{12}$$

where the set $\{\Psi_j\}$, $j = 0 \dots \infty$, is a set of orthogonal polynomials in ξ_k, and $k = 0 \dots \infty$. Truncating terms in equation (11) and substituting equation (12):

$$\left(\sum_{i=0}^{M} \mathbf{K}_i \xi_i(\theta) \right) \left(\sum_{i=0}^{P-1} \mathbf{u}_j \Psi_j(\theta) \right) = \mathbf{f} \tag{13}$$

A more convenient form of equation (13) can be found by minimizing the residual due to truncation, arriving at the following form:

$$\begin{bmatrix} \mathbf{K}_{0,0} & \dots & \mathbf{K}_{0,P-1} \\ \vdots & \ddots & \vdots \\ \mathbf{K}_{P-1,0} & \dots & \mathbf{K}_{P-1,P-1} \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \\ \vdots \\ \mathbf{u}_{P-1} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_0 \\ \vdots \\ \mathbf{f}_{P-1} \end{bmatrix} \tag{14}$$

where $\mathbf{K}_{i,j}$ is an $N \times N$ matrix, $\mathbf{u}_i$ are $N \times 1$ vectors associated with the polynomial expansion of the response, and $\mathbf{f}_i$ are $N \times 1$ vectors of loading. Note that the system to be inverted is $NP \times NP$ in size, so that the size of the PCE expansion will have a significant impact on the computational cost of the solution. The details of this and other derivations can be found in [14].

5.3 Stochastic finite element method for uncertainty propagation in topology optimization

The robust form of the compliance objective function is commonly expressed as the weighted sum of the two statistical measures, namely the mean and standard deviation:

$$\min_{\mathbf{x}} \hat{C} = E[C] + \alpha\sqrt{Var[C]} \tag{15}$$

where $E[C]$ is the expected value of the compliance, $Var[C]$ the variance of the compliance and α is a weighting coefficient chosen by the user. If the loading is deterministic the mean value (expectancy) of the compliance is given by:

$$E[C] = E[\mathbf{f}^\top \mathbf{u}] = \mathbf{f}_0^\top E[\mathbf{u}] \tag{16}$$

In the case of a PCE of the response $E[\mathbf{u}] = \mathbf{u}_0$, where $\mathbf{u}_0$ corresponds to the nodal displacements for polynomial Ψ_0. Finally:

$$E[C] = \mathbf{f}_0^\top \mathbf{u}_0 \tag{17}$$

Once again considering deterministic loading, the variance of the compliance can be found:

$$Var[C] = Var[\mathbf{f}^\top \mathbf{u}] = \mathbf{f}_0^\top Cov[\mathbf{u}]\mathbf{f}_0 \tag{18}$$

where $Cov[\mathbf{u}]$ is the covariance matrix of $\mathbf{u}$, and is found by the expression [14]:

$$Cov[\mathbf{u}] = \sum_{j=1}^{P-1} E[\Psi_j^2]\mathbf{u}_j.\mathbf{u}_j^\top \tag{19}$$

where Ψ_j are the components of the polynomial basis of the displacement field corresponding to displacement vectors $\mathbf{u}_j$. The objective function can then be expressed simply as:

$$\hat{C} = \mathbf{f}_0^\top \mathbf{u}_0 + \alpha \mathbf{f}_0^\top \left(\sum_{j=1}^{P-1} E[\Psi_i^2]\mathbf{u}_j^\top \mathbf{u}_j \right) \mathbf{f}_0 \tag{20}$$

The sensitivities of the objective function with respect to the design variables $\mathbf{x}$ are found making use of the adjoint method, starting from equation (15) and taking the derivative with respect to the design variables (15) as in [11]:

$$\frac{\partial \hat{C}}{\partial \mathbf{x}} = \frac{\partial E[C]}{\partial \mathbf{x}} + \alpha \frac{\partial(\sqrt{Var[C]})}{\partial \mathbf{x}} \tag{21}$$

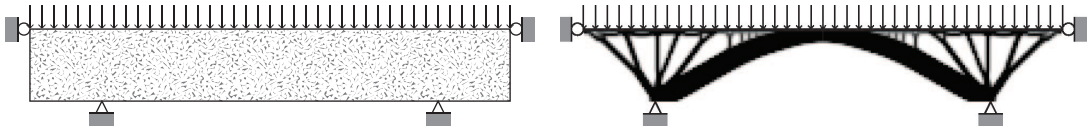
The sensitivities at the element level, as prescribed by the SIMP method, can then be found as follows:

$$\begin{aligned} \frac{\partial \hat{C}_e}{\partial x_e} &= -p x_e^{p-1} \left(\sum_{k=0}^{P-1} \sum_{l=0}^{P-1} \sum_{i=0}^{M} E[\xi_i \Psi_k \Psi_l] \mathbf{u}_{e,k}^\top \mathbf{K}_e^* \mathbf{u}_{e,l} \right) \\ &- \frac{\alpha p x_e^{p-1}}{\sqrt{Var[C]}} \sum_{j=1}^{P-1} \left(\sum_{k=0}^{P-1} \sum_{l=0}^{P-1} \sum_{i=0}^{M} E[\xi_i \Psi_j \Psi_k \Psi_l] \mathbf{u}_{e,k}^\top \mathbf{K}_e^* \mathbf{u}_{e,l} \right) \mathbf{u}_j^\top \mathbf{f}_0 \end{aligned} \tag{22}$$

The above expression is very similar to the expression for the displacement constraints as found by [13].

6. Computational examples

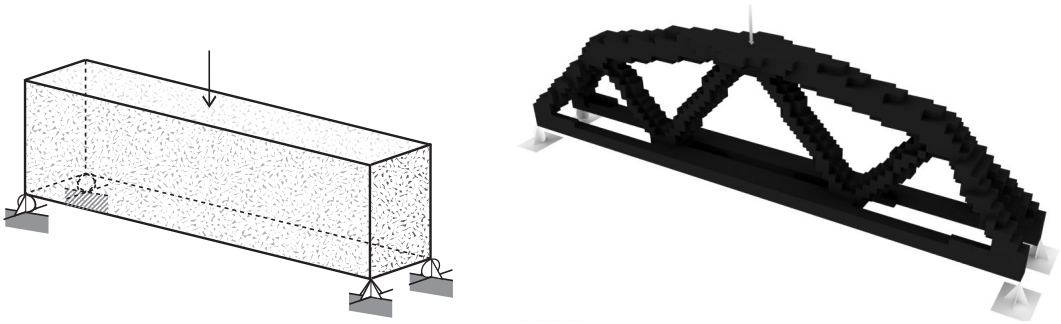
The proposed method is demonstrated on both 2D and 3D continuum and truss problems. A 2D bridge problem is considered, in which the effects of the variation of the material parameters are shown. The domain is discretized using 2D quad elements. Next a 3D bridge structure is considered, for which 8-node brick elements are used. Finally a truss problem demonstrates the approach to truss optimization. Results are shown for various values of the standard deviation and correlation length of the random field, as well as the additive coefficient in the objective function. The expression in (22) has been rigorously tested using finite difference checks to confirm their validity in the following examples, showing excellent accuracy. For all computational examples the order of the random field expansion is $M = 2$. Similarly the order of the polynomial chaos expansion of the responses is taken to be $P = 2$. The method of moving asymptotes [22] was used to solve the optimization problem in the computational examples.



(a) The domain, boundaries and loading of the 2D bridge problem. The uncertainties on material properties of the bulk are taken into account.

(b) The deterministic solution to the 2D bridge problem.

Figure 2: 2D bridge problem: Problem set up and deterministic solution.



(a) The domain, boundaries and loading of the 3D bridge problem. The uncertainties on material properties of the bulk are taken into account.

(b) Deterministic solution to the 3D bridge problem.

Figure 3: 23 bridge problem: Problem set up and deterministic solution.

6.1 2D continuum bridge

6.1.1 Problem

In this problem a 2D bridge problem is considered. For this purpose symmetry is taken into account and the domain is discretized using 100×30 2D quad elements. The problem domain, supports and loading are shown in figure 2(a). Vertical unit loads are applied to each node on the top edge of the structure.

6.1.2 Results

The deterministic solution to this problem is obtained by setting the standard deviation to zero and is shown in figure 2(b). In table 1 an overview of the solutions to the problem are given, for values of σ between 0.2 and 0.6, $\alpha = 2$ and $\alpha = 4$, and correlation length l between 5 and 20. Clear differences in the topology are visible for various values of the parameters. However, the overall form, a single arch with secondary supporting struts, is a feature of all of the solutions. Along with the topology of the struts, the shape of the arch also varies.

6.2 3D Continuum bridge

6.2.1 Problem

In this example the 3D domain is defined as shown in figure 3(a). The structural domain is discretized using $80 \times 10 \times 20$ 8-node brick elements, loaded by a single unit point load in the center of the top face, and is simply supported at the 4 lower corners. Symmetry is used to reduce the problem size. For all of the solutions that follow, the volume fraction is 0.2, $r_{min} = 1.5$, $\alpha = 3$, $\mu = 1$ and $p = 3$ penalization is used. The standard deviation and correlation lengths are varied.

6.2.2 Results

The deterministic solution is shown in figure 3(b), while the resulting topologies for various values of the parameters σ, α, and l are shown in table 2. For small values of σ only minor variation is visible, however, as can be expected, for larger values distinctively different solutions are found. Note that for all the non-deterministic solutions the main arch of the structure is split in two meeting only at the loading

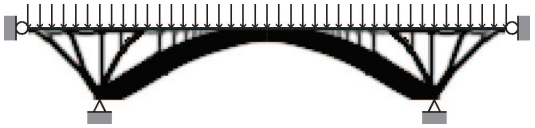
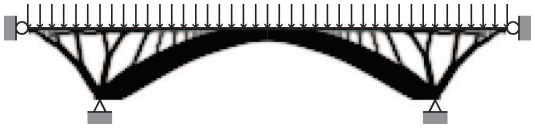
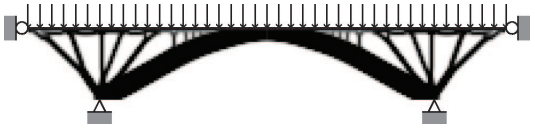
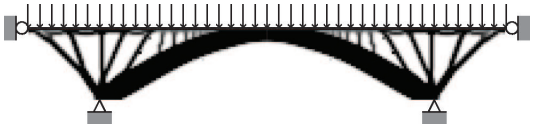
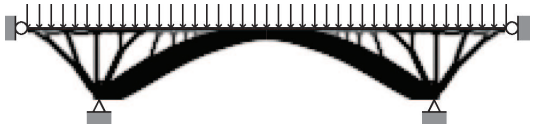
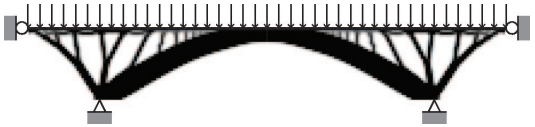
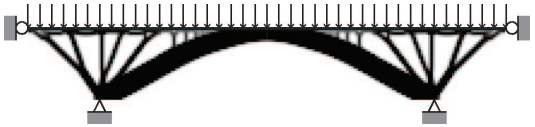
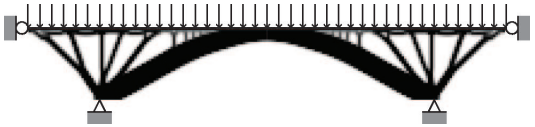
σ	α	$l = 5$	$l = 15$	$l = 20$
0.2	2			
	4			
0.4	2			
	4			
0.6	2			
	4			

Table 1: Resulting topologies for various values of the standard deviation σ, the factor α and the correlation length l.

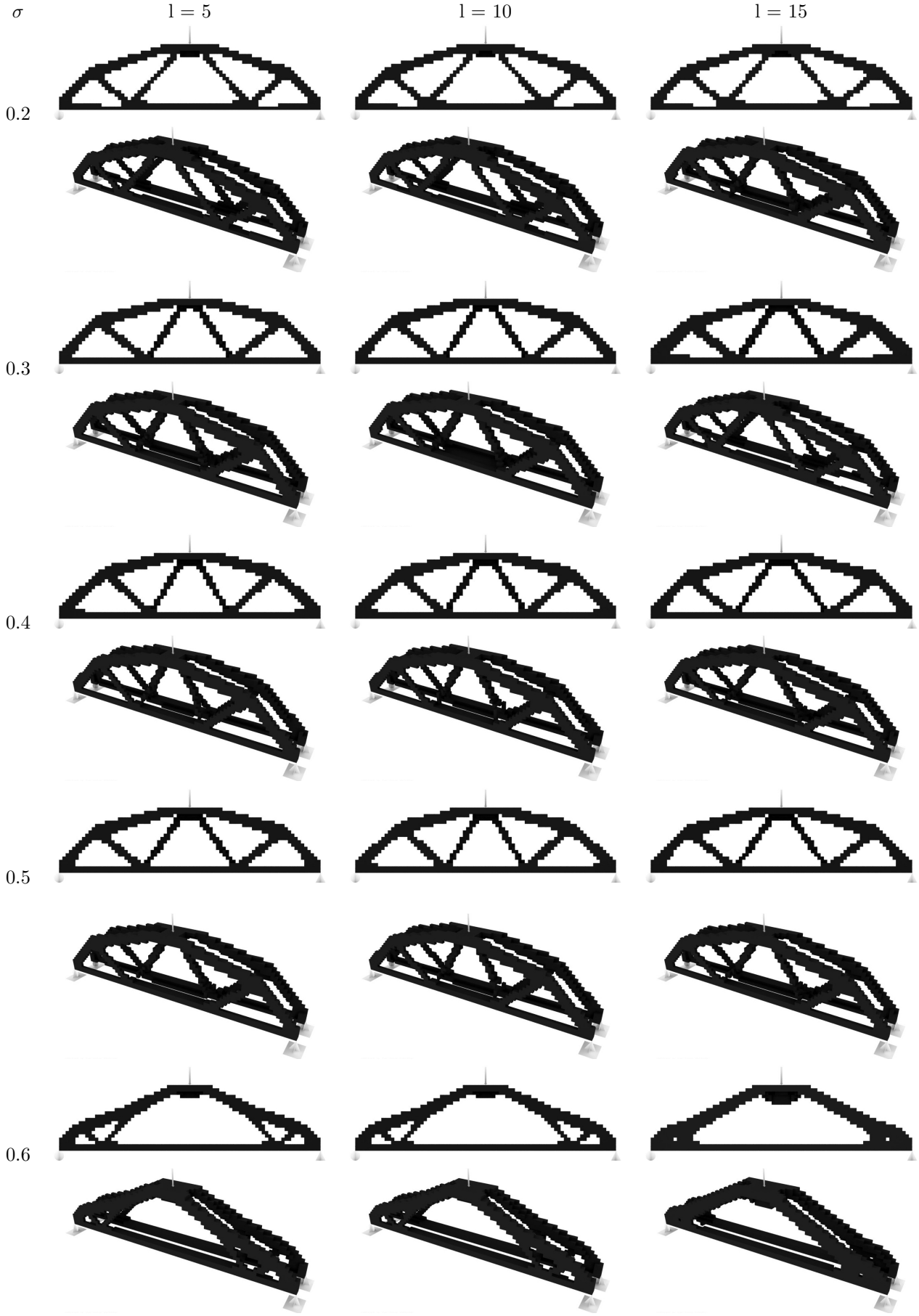
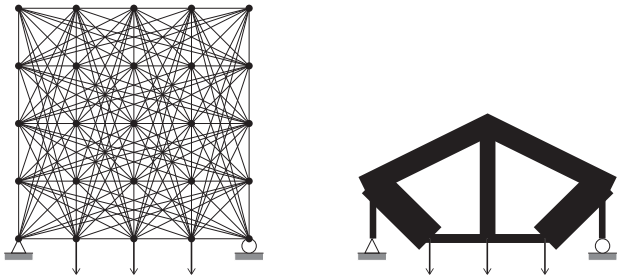


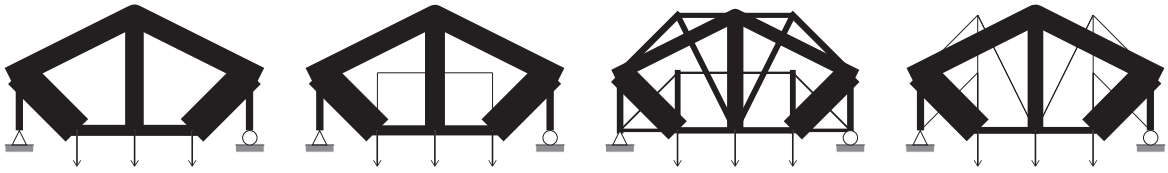
Table 2: Resulting topologies for various values of the standard deviation σ and the correlation length l.



(a) 2D truss problem ground structure

(b) 2D truss problem deterministic solution

Figure 4: 2D truss problem: Problem set up and deterministic solution.

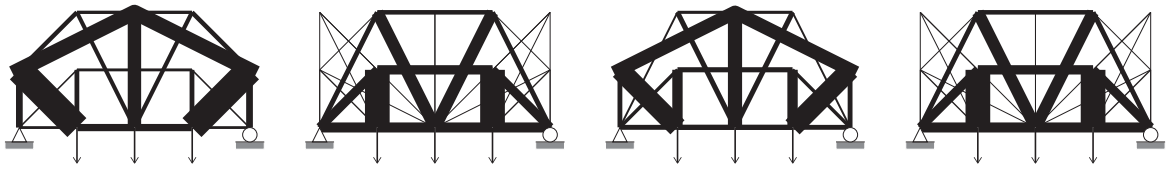


(a) Solution $\sigma = 0.1$, $\alpha = 5$, and $l = 0.5$

(b) Solution $\sigma = 0.1$, $\alpha = 10$, and $l = 0.5$

(c) Solution $\sigma = 0.2$, $\alpha = 5$, and $l = 0.5$

(d) Solution $\sigma = 0.2$, $\alpha = 10$, and $l = 0.5$



(e) Solution $\sigma = 0.4$, $\alpha = 5$, and $l = 0.5$

(f) Solution $\sigma = 0.4$, $\alpha = 10$, and $l = 0.5$

(g) Solution $\sigma = 0.5$, $\alpha = 5$, and $l = 0.5$

(h) Solution $\sigma = 0.5$, $\alpha = 10$, and $l = 0.5$

Figure 5: 2D truss problem deterministic solution

point, while a single arch characterizes the deterministic solution. Another prominent difference is the topology of the middle struts in the truss-like continua resulting from the optimization process. In several of the structures these struts are separate on either side of the bridge deck, however, with higher values of σ the two struts merge into one, splitting apart near the bottom. For values of $\sigma = 0.6$ the structures change radically, especially for higher values of the correlation length l. In profile it can be seen that the general shape of the arch is more rounded for higher values of l.

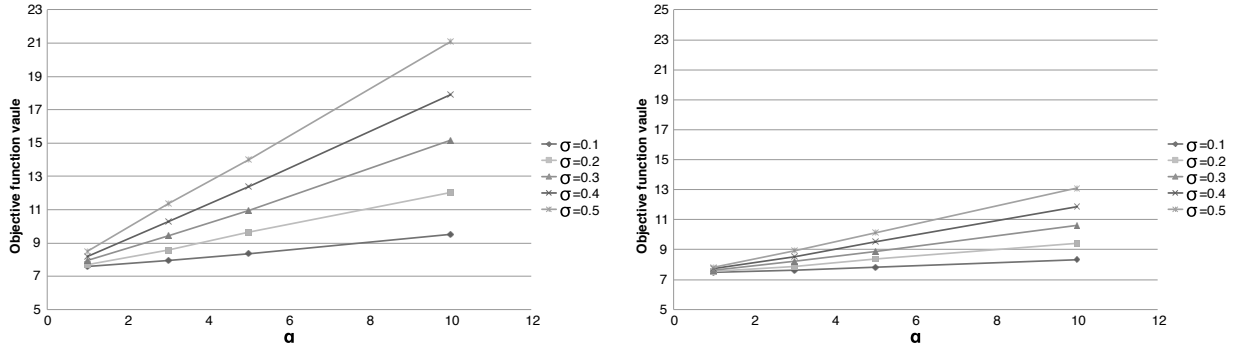
6.3 2D truss problem

6.3.1 Problem

A 2D truss problem found in [7] is used to demonstrate the method. This problem consists of 25 nodes, each connected to every other node by a bar element, 300 elements in total. The structure is simply supported at two bottom corner nodes and loaded along the bottom edge of the structure by unit vertical loads (figure 4(a)). The elements are subdivided into 5 segments and a volume fraction of 1.25 is taken.

6.3.2 Results

The deterministic solution to the problem is shown in figure 4(a). The solutions for various values of the standard deviation and α-coefficient are given in figure 5. The resulting structures differ quite significantly from those found in the reference publication, where the perturbation method was used and one variable associated with each element. The values of the objective functions are plotted for various values of the α and σ, for $l = 0.5$ in figure 6(a). A linear relation can be seen between the various solutions, as expected. If these solutions are compared to the same plot for $l = 0.1$ (figure 6(b)), it can clearly be seen that the value of the correlation length has an effect on the dispersion of the solutions.



(a) results for correlation length $l = 0.5$, for various values of σ and α

(b) results for correlation length $l = 0.1$, for various values of σ and α

Figure 6: Values of the objective function for various parameters

7. Conclusions

This research presents a framework for topology optimization of both continuum and truss structures using spectral stochastic finite element method. A novel approach to truss analysis is introduced to model material uncertainties across elements of varying dimensions. The method is demonstrated on both 2D and 3D continuum examples and on a truss example. Further work is necessary to investigate the full effect of the variation of the parameters.

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