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Adaptive Quasi Static Ritz Vector (AQSRV)-based Model Reduction Scheme for the Numerical Simulation of Broadband Acoustic Metamaterials

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1. Abstract
This paper suggests the adaptive quasi static Ritz vector (AQSRV)-based model reduction scheme of applying an error indicator for efficient numerical simulation of the acoustic metamaterial systems with the characteristics of broadband, inhomogeneous, and anisotropic material properties. In contrast to a conventional mode displacement method (MDM)-based model reduction scheme, the proposed method has characteristics of the target expansion frequency, multiple subintervals, and error indicator. Moreover, since AQSRV-based model reduction scheme considers mass matrix, stiffness matrix, and load vector simultaneously in order to obtain the basis vectors, the accuracy and efficiency of overall analysis exponentially increase. In order to model the inhomogeneous and anisotropic acoustic metamaterial systems, this study uses acoustic Helmholtz equation as the governing equation, and then it is discretized by standard finite element method. The proposed method is applied to the various numerical applications such as the 2D simple lateral duct with the various material characteristics, the zero index metamaterial (ZIM)-based omnidirectional speakers, the gradient index acoustic lens, and acoustic cloaking. Through these numerical examples, the performance of proposed AQSRV-based model reduction scheme is verified in terms of accuracy and computational efficiency. These verification examples shows that the proposed AQSRV-based model reduction scheme can be used for analysis and design of the future various acoustic metamaterial systems.

2. Keywords: Acoustic metamaterials, Adaptive quasi static Ritz vector (AQSRV), Error indicator, and Multiple subintervals

3. Introduction
Acoustic metamaterials are artificial material with effective acoustic properties (e.g., negative mass density and negative bulk modulus) not found in constituent material and not readily observed in nature. Recently, the acoustic metamaterials are widely used as words indicating the every system for wave control by the artificially engineered materials. The typical applications of acoustic metamaterial are acoustic lens, acoustic cloaking, etc. Especially, the acoustic lens is based on the typical feature of acoustic metamaterials, negative refraction. The feature of negative refraction is shown well in Figure 1.
For future analysis and design related to acoustic metamaterials, the study for efficient and robust computational method should be preceded. Also, the numerical analysis will play an important role in acoustic metamaterial research. Therefore, the development of efficient and robust numerical simulation technique is very significant factor in order to go one step further.
For numerical analysis of acoustic metamaterial systems, most researchers use the FDTD (Finite Difference Time Domain method) and/or FEM (Finite Element Method). Especially, FEM is more used. The reason is that FEM is relatively simple to apply the inhomogeneous and anisotropic material characteristics, compared with the other numerical methods. Moreover, FEM is easily applied to complex geometry. However, the computational cost is very high and the efficient management of computer memory is difficult for the FEM. These disadvantages stand out even more for broadband and large scale systems.
Therefore, this study presents the AQSRV-based model reduction scheme in order to deal with the inhomogeneous and anisotropic characteristics of the acoustic metamaterials. In addition, the proposed AQSRV-based model reduction scheme has the two advantages of the multiple subintervals and the adaptive selection of the basis vector corresponding to each subinterval through the error indicator. This method gives you the opportunity to perform the analysis and design of the acoustic metamaterial systems more efficiently.
In this respect, the main contributions of this study are classified into the 4 parts: first of all, we first applied the model reduction scheme to the numerical analysis of the broadband and the large-scale acoustic metamaterial systems. Second, the model reduction scheme considering the inhomogeneous and anisotropic characteristics is suggested. Third, adaptive selection of the basis vector corresponding to each subinterval by considering error indicator. This characteristic will play an important role in design optimization (e.g., topology optimization). The reason is that the internal structure of system matrix is always changed in each design step. Finally, we applied the

proposed AQSRV-based model reduction scheme to the various acoustic metamaterial systems (e.g., omnidirectional speaker, acoustic lens, acoustic cloaking, etc.).

The remainder of this paper is organized as follows. Section 4 explains the governing equation and boundary conditions for acoustic metamaterial system modeling, and builds a numerical model using the finite element method and Galerkin method. Section 5 then presents several characteristics and advantages of the proposed AQSRV-based model reduction scheme. In addition, algorithm to obtain basis vector (i.e., AQSRV) set is given. Next, Section 6 presents several numerical examples based on the proposed model reduction scheme, and shows that the method can be successfully applied to the numerical analysis of acoustic metamaterial systems. In Section 7 we draw the conclusions and discuss findings of the study.

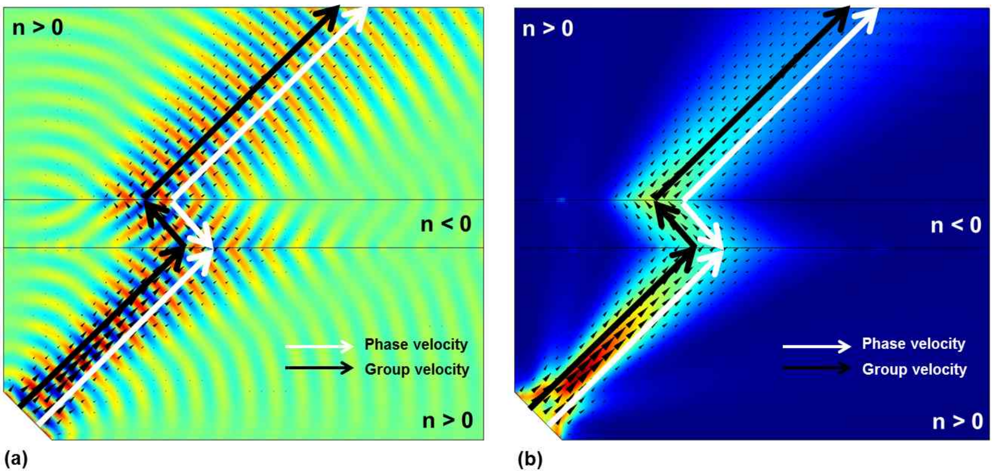

Figure 1: Typical features of the acoustic metamaterial: negative refraction; (a) Sound pressure for the 2500Hz plane acoustic wave, and (b) Sound intensity for the 2500Hz plane acoustic wave

4. Modeling of the Acoustic Metamaterials based on the Finite Element Formulation

In this section, the governing equation, boundary conditions, and finite element model are defined prior to a numerical simulation of the acoustic metamaterial systems. In addition, finite element formulation is used in algorithm to obtain the basis vectors of the proposed AQSRV-based model reduction scheme. Note that a more detailed explanation of this algorithm will be given in Section 5.

4.1. Governing Equation and Boundary Conditions

In general, the acoustic metamaterials have the characteristics of inhomogeneous and anisotropic material properties (e.g., mass density, $\boldsymbol{\rho}$, and bulk modulus, κ). Therefore, we should use the inhomogeneous and anisotropic Helmholtz equation as the governing equation in order to model the acoustic metamaterial systems. The Helmholtz equation is expressed as Eq. (1). Then, various boundary conditions are considered based on expressions form Eq. (2)-(4). The boundary conditions can be classified as either Neumann or Dirichlet type; the conceptual domain and boundaries being considered for the acoustic metamaterial system modeling appear in Figure 2.

Inhomogeneous and anisotropic Helmholtz equation

$$\nabla\cdot\left(\boldsymbol{\rho}^{-1}\cdot\nabla p\right)+\frac{\omega^2}{\kappa}p=Q \quad in\ \Omega_{Acoustic} \quad where,\ \left\{\boldsymbol{\rho}=\boldsymbol{\rho}(\mathbf{r})=\begin{bmatrix}\rho_{xx} & \rho_{xy} & \rho_{xz}\\ \rho_{yx} & \rho_{yy} & \rho_{yz}\\ \rho_{zx} & \rho_{zy} & \rho_{zz}\end{bmatrix},\ \kappa=\kappa(\mathbf{r})\right. \tag{1}$$

Imposed pressure boundary condition (Dirichlet type)

$$p=p_0 \quad on\ \partial\Omega_{DP} \tag{2}$$

Imposed normal velocity boundary condition (Neumann type)

$$\mathbf{n}\cdot\left(\boldsymbol{\rho}^{-1}\cdot\nabla p\right)=G=-j\omega v_n \quad on\ \partial\Omega_{NV} \tag{3}$$

Imposed normal impedance boundary condition (Neumann type)

$$\mathbf{n}\cdot\left(\boldsymbol{\rho}^{-1}\cdot\nabla p\right)=G=-\frac{jp\omega}{\mathbf{Z}} \qquad on\ \partial\Omega_{NI} \tag{4}$$

where pressure in the acoustic domain is denoted by p. The $\boldsymbol{\rho}$, κ are the mass density and bulk modulus, respectively. The Q, v_n, and $\mathbf{Z}$ are the external source (e.g., monopole), the normal velocity, and the surface impedance, respectively.

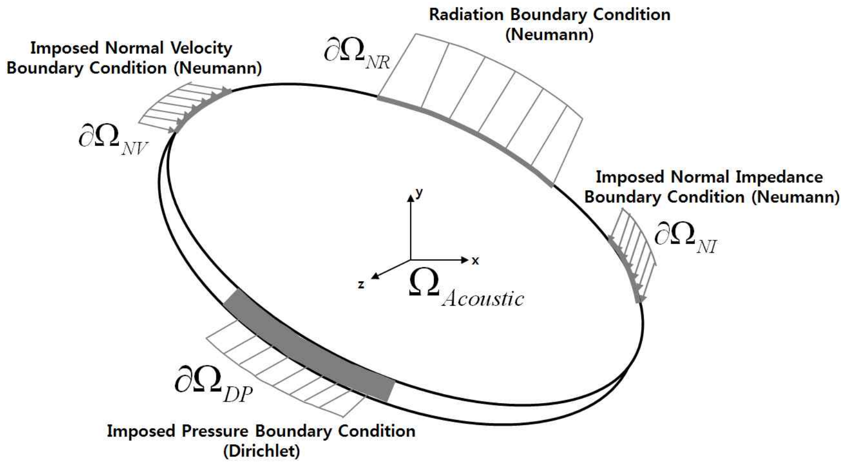

Figure 2: Conceptual system domain and boundaries for acoustic metamaterial system modeling

4.2. Finite Element Model

The inhomogeneous and anisotropic Helmholtz equation is discretized using the standard Galerkin method. After obtaining variational equations of the governing equation for acoustic metamaterial, algebraic equation in the form of a matrix are obtained. The discretization process is performed by the standard quadratic Lagrange shape function. After multiplying the variation of a pressure on both sides of the governing equations, the variational equation of the Helmholtz equation can be obtained using the divergence theorem.

$$\int_{\Omega_{Acoustic}} \left(\nabla\delta p\right)\cdot\left(\boldsymbol{\rho}^{-1}\cdot\nabla p\right)+\delta p\left(Q-\frac{\omega^2}{\kappa}p\right)d\Omega-\int_{\partial\Omega}\left(\delta p\right)GdS=0 \tag{5}$$

where variation of the pressure are denoted by δp. A standard discretization of the variational equation, Eq. (5), yields the following linear algebraic equation to be solved (for more details, the reader is referred to [1]).

$$\mathbf{SP}=\mathbf{F} \quad where, \begin{cases}\mathbf{S}=-\omega^2\mathbf{M}+j\omega\mathbf{C}+\mathbf{K}\\ \mathbf{F}=\mathbf{F}_G-\mathbf{F}_Q\end{cases} \tag{6}$$

where the nodal pressure vectors are denoted by **P**. The stiffness matrix, damping matrix, mass matrix, and external force vector are denoted by **K**, **C**, **M**, and **F**, respectively. The fully combined system matrix is denoted by **S**. **K** corresponds to the first term of Eq. (5), **M** corresponds to the second term of Eq. (5), **C** corresponds to the third term of Eq. (5), and **F** corresponds to the third term of Eq. (5) or boundary conditions, Eq. (3)-(4). Commercial software (COMSOL multiphysics) and Matlab-based programming are employed for all finite element procedures [2].

5. Proposed the AQSRV-based Model Reduction Scheme

The fundamental principle of the model reduction scheme is very simple. That is, the original large-scale system is reduced to the small-scale system using basis vector set. According to the type of basis vectors, there are a number of model reduction scheme. Thus, the most important is how to select what kinds of the basis vectors. Based on these facts, we propose the adaptive quasi static Ritz vector (AQSRV) in this paper.

5.1. Characteristics of the AQSRV-based Model Reduction Scheme

Compared with the conventional model reduction scheme, the proposed AQSRV-based model reduction scheme has the two main characteristics. The first is multiple subintervals (i.e., multiple expansion frequencies). The second is adaptive selection of the basis vector (i.e., AQSRV) through the error indicator.

Above all, let us see the first characteristic (i.e., multiple subintervals). In the Figure (3a), the frequency interval is just one (i.e., ω = 0 ~ 1.5 rad/s). This means the expansion frequency is also one. As you can see the figure, the frequency response is very accurate near the expansion frequency. However, the accuracy of frequency response is decreased, getting further and further away from the expansion frequency. In the Figure (3b), the frequency subintervals are three (i.e., subinterval.1: ω = 0 ~ 0.5 rad/s, subinterval.2: ω = 0.5 ~ 1.0 rad/s, and subinterval.3: ω = 1.0 ~ 1.5 rad/s). Compared with the Figure (3a), the overall frequency response is very accurate. The key point here is that the number of basis vectors used is the same in the case of Figure (3a). The result clearly shows that the feature of multiple subintervals further improves the accuracy of numerical analysis.

Next, the proposed AQSRV-based model reduction scheme has the characteristic of the adaptive selection of basis vector through the error indicator. That is, in order to adaptively obtain the basis vectors for each subinterval, the error indicator is employed in the process of determining the basis vectors. The adaptive selection of basis vector means that each basis vector corresponding to each subinterval can have different number, as shown Figure (3b). Through this adaptive selection of basis vector, the efficiency and accuracy of the numerical analysis is further improved.

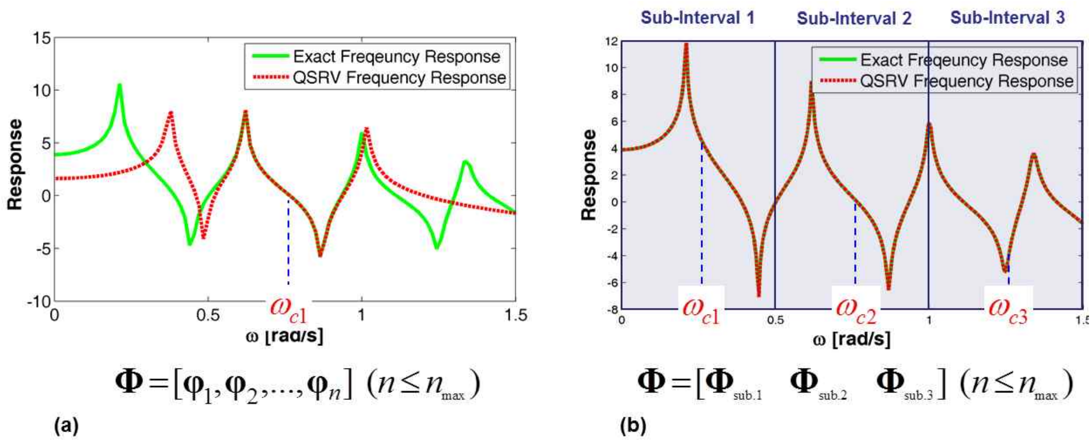

Figure 3: Comparison between the frequency responses with respect to the number of subintervals and the number of basis vector corresponding to each subinterval; (a) the frequency response when the number of subintervals is one, and (b) the frequency response when the number of subintervals is three, and the number of basis vector is different each other for each subinterval

5.2. The Algorithm and Flow Charts for the AQSRV-based Model Reduction Scheme

As mentioned above, compared with the conventional model reduction scheme (e.g., MDM), the AQSRV-based model reduction scheme has difference for setting expansion frequency. Moreover, the accuracy of numerical analysis using the AQSRV-based reduced model is more accurate than using the MDM-based reduced model. The reason is that the algorithm to obtain basis vector for the AQSRV-based model reduction scheme considers a mass, stiffness, external force, and expansion frequency simultaneously, contrary to the MDM-based model reduction scheme. For the MDM-based model reduction scheme, basis vector is the same as eigenvector [3]. As you already know, the eigenvector can be obtained from the eigenvalue problem ignoring the influence of the external force. The algorithm to obtain basis vector for the AQSRV-based model reduction scheme is as followings, Eq. (7)-(12). The overall procedure of this algorithm is very similar to the [4] with the exception of the error indicator part equations, Eq. (10)-(12).

Solving for the primary static vectors

$$[\mathbf{K} - \omega_c^2 \mathbf{M}]\boldsymbol{\varphi}_j^* = \mathbf{F}_j, \quad where \quad \begin{cases} \mathbf{F}_j = \mathbf{F}(\mathbf{s}) & for \quad j = 1 \\ \mathbf{F}_j = \mathbf{M}\boldsymbol{\varphi}_{j\text{-}1} & for \quad j > 1 \end{cases} \tag{7}$$

Orthogonalizing with respect to the previous adaptive quasi static Ritz vectors

$$\boldsymbol{\varphi}_j^{**} = \boldsymbol{\varphi}_j^* - \sum_{k=1}^{j-1} \alpha_{jk}\boldsymbol{\varphi}_k, \quad where \quad \begin{cases} \alpha_{jk} = 0 & for \quad j = 1 \\ \alpha_{jk} = \boldsymbol{\varphi}_j^{*T}\mathbf{M}\boldsymbol{\varphi}_k & for \quad j > 1 \end{cases} \tag{8}$$

Normalizing with respect to **M** *matrix*

$$\boldsymbol{\varphi}_k = \beta_j \boldsymbol{\varphi}_j^{**}, \quad where \quad \beta_j = \left(\sqrt{\boldsymbol{\varphi}_j^{**T}\mathbf{M}\boldsymbol{\varphi}_j^{**}}\right)^{-1} \tag{9}$$

Solving the reduced system using current obtained basis vector set

$$\left[-\omega_r^2\mathbf{M}_r^{\ 2} + j\omega_r\mathbf{C}_r + \mathbf{K}_r\right]\mathbf{Q} = \mathbf{F}_r, \qquad where \quad \begin{cases}\mathbf{P} \cong \mathbf{P}_A = \boldsymbol{\Phi}\mathbf{Q} \\ \boldsymbol{\Phi} = [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \cdots, \boldsymbol{\varphi}_n]\end{cases} \tag{10}$$

Error indicator

$$\varepsilon_j = \frac{\left\|\mathbf{J}_j - \mathbf{J}_{j-1}\right\|}{\left\|\mathbf{J}_j\right\|} \times 100 \tag{11}$$

Convergence test

$$If \ \left(\varepsilon_j < \varepsilon_{specified}\right) \ \rightarrow \ \textbf{break} \tag{12}$$

where ω_c is expansion frequency, **Q** is reduced state variable vector, **Φ** is projection matrix composed of the basis vector set. In Eq. (11), ε_j is error indicator at j-th iterations. $\mathbf{J}_j$ can be any type of error indicator functions, Selection of the suitable type of error indicator function is important. In general, the frequency response function is selected as the error indicator function. The basis vector (i.e., AQSRV) can be obtained efficiently using this developed algorithm. By multiplying projection matrix (i.e., **Φ**) composed of these obtained basis vectors on both sides of Eq. (6), the reduced model can be obtained. This process is well described in the Figure 4.

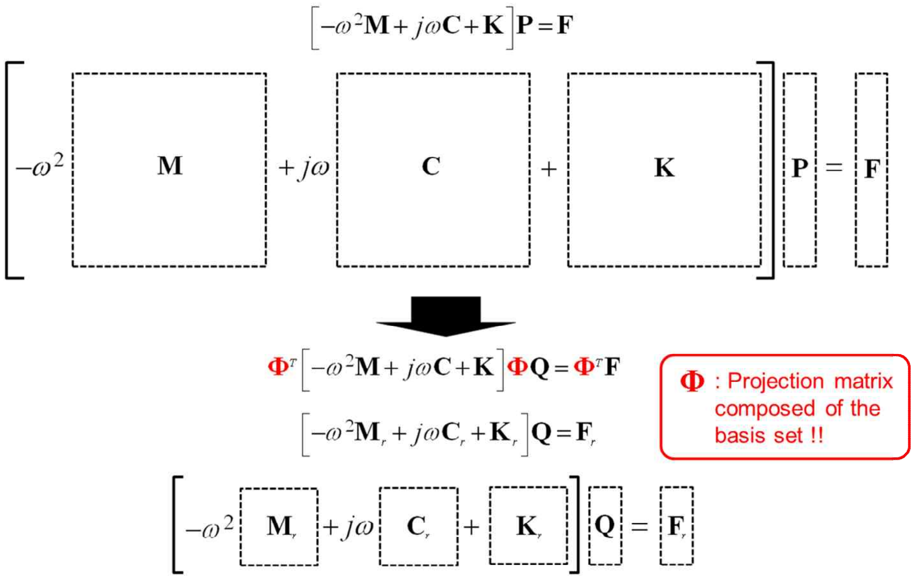

Figure 4: System dimension reduction using the proposed AQSRV-based model reduction scheme

6. Applications

So far we have explained the proposed AQSRV-based model reduction scheme. In this section, the proposed AQSRV-based model reduction scheme is applied to the two numerical examples like 2D simple lateral duct. Through these numerical, the performance of proposed model reduction scheme is verified from the accuracy and computational cost perspective. Moreover, we investigate aforementioned two main characteristics of the proposed AQSRV-based model reduction scheme, such as the multiple subintervals and the adaptive selection of basis vectors.

6.1. Numerical example.1: 2D Simple Lateral Duct with Zero-Index Metamaterials

The first numerical example is 2D simple lateral duct with zero index metamaterials. Zero-index metamaterial means that the mass density has near zero value and the bulk modulus has very large value compared with background bulk modulus. The typical characteristics of the zero-index metamaterial are total transmission and total reflection with no loss. Therefore, by using these characteristics of the zero-index metamaterials, the various acoustic applications can be explored. The author mainly focuses on the omnidirectional loudspeaker using the zero-index metamaterials. The numerical example for this omnidirectional speaker system will be explained in detail in the following sections. The system geometry configurations and dimensions are shown well in the Figure 5. As you can see the figure, the zero-index metamaterial is placed in middle position of the duct.

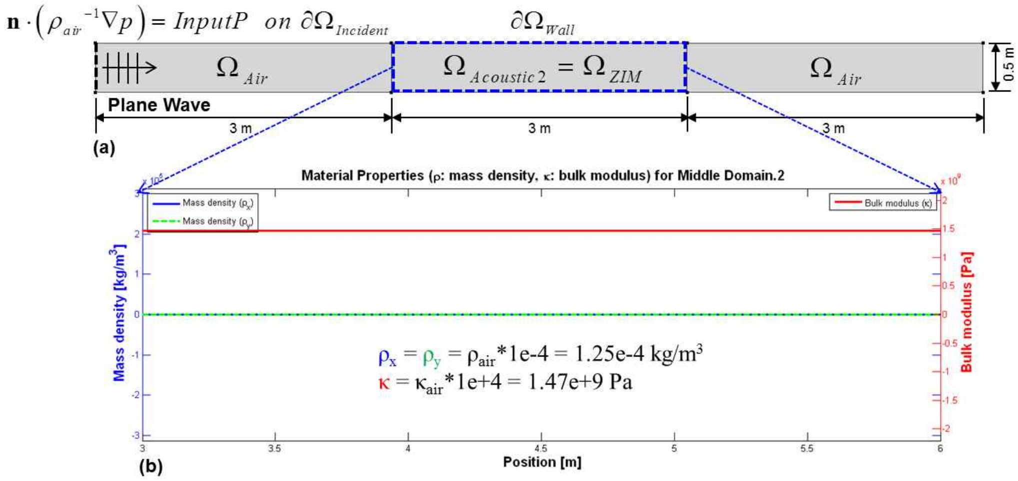

Figure 5: Numerical example.1: (a) System configuration, boundary condition, material distributions, and dimensions of the 2D simple lateral duct with zero index metamaterials, and (b) zero-index metamaterial has the material characteristic of a near zero mass density and very large bulk modulus

Through this numerical example, the verification of whether proposed AQSRV-based model reduction scheme is well applied is performed in terms of accuracy and computational time. In addition, the characteristic of total transmission with no loss of zero-index metamaterials is investigated. The frequency range of interest is 100-1000Hz, and the number of subinterval is 10.

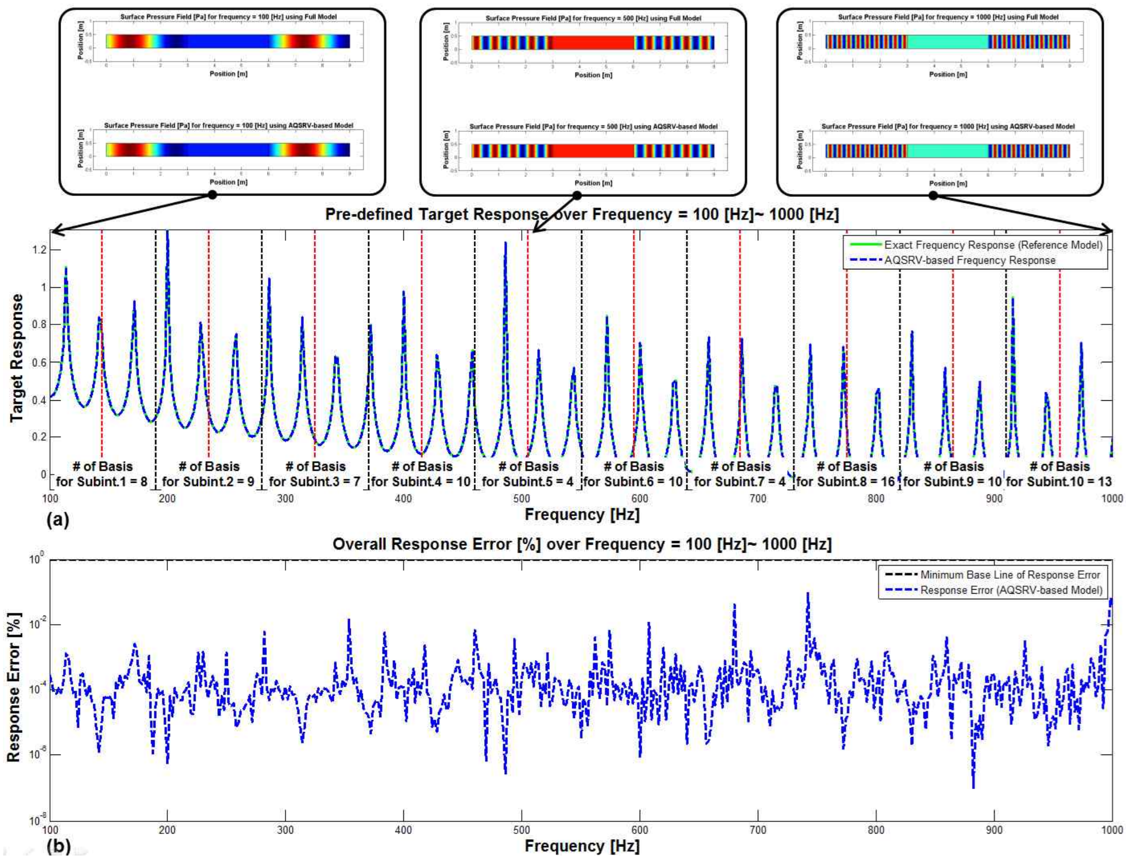

Figure 6: The performance for the full model and the reduced model of the numerical example.1; (a) predefined target response over the frequency band from 100 to 1000 Hz, and (b) response error between the full model and reduced model over the frequency band from 100 to 1000 Hz

Table 1: Comparison between the full and reduced model of the numerical example.1, in terms of accuracy and computational time

	Computation time (sec)	The number of basis vector	Efficiency (Time)	Efficiency (Accuracy)
Full model	572.6	3500	572.6/572.6 =1	
Reduced model	2.82	91	572.6/2.82 = 203	Good (error is very small)

Figure 6 shows the performance for the full model and the reduced model. First of all, the predefined target frequency responses between the full model and the reduced model are in good agreement with each other as shown in the Figure (6a). In order to analyze more closely, we present the frequency response error as shown in Figure (6b). The frequency response error is calculated by Eq. (11). As you can see the graph, the error is very small. This degree of error can be accepted sufficiently from the analysis and design perspective. Furthermore, the system matrix size of the reduced model is 91 by 91. This size is decreased about 40 times compared to the size of the original. Thus, the computational speed is greatly improved about 200 times.

In addition, the characteristic of total transmission with no loss is shown well in upper of the Figure (6a). The total transmission characteristic is almost the same for both of numerical results from the full model and the reduced model. Therefore, the performance of the proposed AQSRV-based model reduction scheme is verified from these results.

6.2. Numerical example.2: 2D Simple Lateral Duct with Inhomogeneous and Anisotropic Metamaterials

The second numerical example is 2D simple lateral duct with inhomogeneous and anisotropic metamaterials. As mentioned before, the typical characteristics of the acoustic metamaterials are inhomogeneous and anisotropic material properties (i.e., mass density, $\boldsymbol{\rho}$, and bulk modulus, κ). The typical acoustic metamaterial system to use these characteristics (i.e., inhomogeneous and anisotropic) is the acoustic cloaking device. The numerical example for this acoustic cloaking device will be explained in detail in the following sections. Through this numerical example, the verification of whether our own proposed model reduction scheme is well applied is carried out in terms of the inhomogeneous and anisotropic material properties characteristics. The frequency range of interest is 100-1000Hz, and the number of subinterval is 10. The system geometry configurations and dimensions are shown well in the Figure .7. The placement of acoustic metamaterials is the same as previous numerical example.

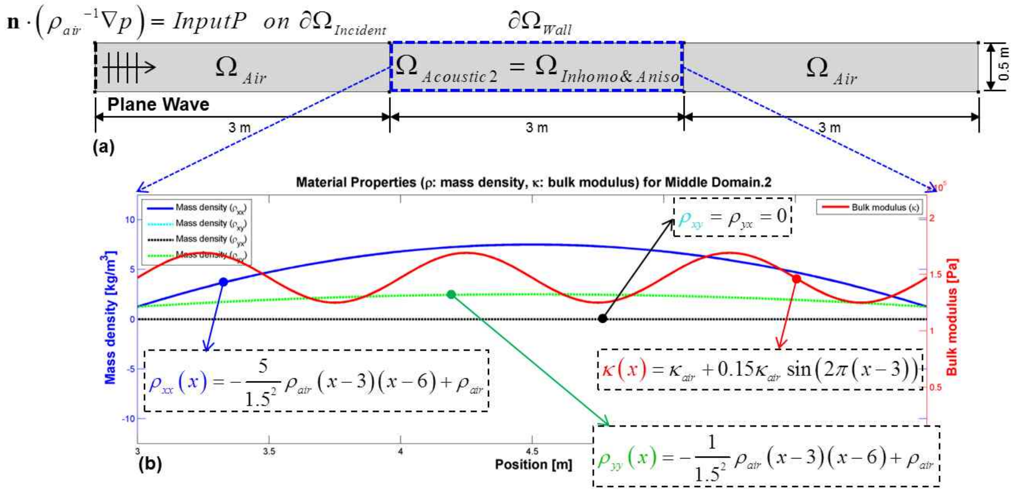

Figure 7: Numerical example.2: (a) System configuration, boundary condition, material distributions, and dimensions of the 2D simple lateral duct with inhomogeneous and anisotropic metamaterials, and (b) The artificial inhomogeneous and anisotropic material properties are assigned. The mass density is modeled by quadratic function profile and the bulk modulus is modeled by sine function profile

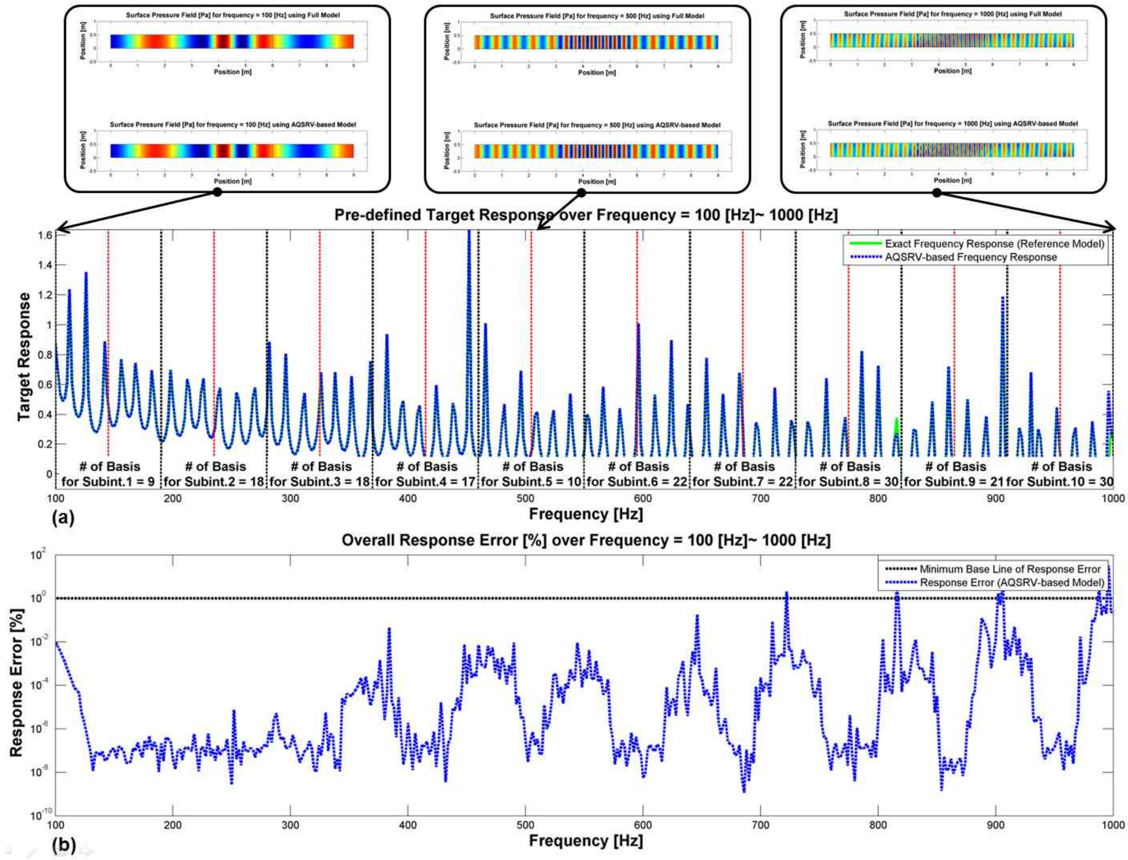

Figure 8: The performance for the full model and the reduced model of the numerical example.2; (a) predefined target response over the frequency band from 100 to 1000 Hz, and (b) response error between the full model and reduced model over the frequency band from 100 to 1000 Hz

Table 2: Comparison between the full and reduced model of the numerical example.2, in terms of accuracy and computational time

	Computation time (sec)	The number of basis vector	Efficiency (Time)	Efficiency (Accuracy)
Full model	545.9	3500	545.9/545.9 =1	
Reduced model	11.42	197	545.9/11.42 = 47.8	Good (error is very small)

Figure 8 shows the performance for the full model and the reduced model. First of all, the predefined target frequency responses between the full model and the reduced model are in good agreement with each other as shown in the Figure (8a). In order to analyze more closely, we present the frequency response error as shown in Figure (8b). The frequency response error is calculated by Eq. (11). As you can see the graph, the error is very small. This degree of error can be accepted sufficiently from the analysis and design perspective. Furthermore, the system matrix size of the reduced model is 197 by 197. This size is decreased about 18 times compared to the size. Thus, we can find that the proposed AQSRV-based model reduction scheme has a good performance from the very complex material property perspective.

7. Conclusions

In this paper, it was shown that the proposed model reduction scheme could be employed to simulate various acoustic metamaterial systems. The model reduction scheme is based on the projection basis vectors. Therefore, the original large scale system matrix is reduced to a much smaller scale. Moreover, the accuracy and efficiency of the simulation is further improved by a characteristic of the error indicator and multiple frequency subintervals. It was subsequently verified by several numerical examples such as 2D simple lateral duct, omnidirectional speaker, acoustic lens, and acoustic cloaking. As mentioned before, the numerical analysis will play an important role in acoustic metamaterial research. Therefore, the development of efficient and robust numerical simulation technique is very significant factor in order to go one step further. The main findings of this study are hereby summarized as follows.

- **a.** The proposed AQSRV-based model reduction scheme is indispensable to the large-scale systems and the dynamic problems.
- **b.** The proposed model reduction scheme has multiple expansion frequencies corresponding to the multiple subintervals. As a result, the accuracy of numerical solution is further improved.
- **c.** The proposed model reduction scheme has different number of basis vector sets for each subinterval through the simple error indicator. Thus, the efficiency of the numerical simulation is improved.
- **d.** The proposed model reduction scheme is even more efficient than the conventional mode displacement method (MDM)-based model reduction scheme, from the accuracy and computational cost perspective.
- **e.** From the several numerical examples, we can know that the proposed model reduction scheme has a more good performance for the inhomogeneous and anisotropic acoustic metamaterials, in comparison with the existing model reduction schemes (e.g., MDM, QSRV [3-4]).

Finally, we expect that the proposed AQSRV-based model reduction scheme described in this study would be a good way to simulate and design the various acoustic metamaterial systems over the broadband operation frequency (e.g., omnidirectional speaker using zero index metamaterial, acoustic cloaking, acoustic lens, etc.).

8. References

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