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A MinMax Framework for Robust Design Optimization

Mattia Padulo[†] and Meng–Sing Liou[‡]

[†]Airbus, 316 Route de Bayonne, 31060 Toulouse CEDEX 9 France. E-mail: mattia.padulo@airbus.com
[‡]NASA Glenn Research Center, Cleveland, OH, 44135, USA

1. Abstract
Presented in the paper is a comprehensive methodology to cast and deploy moment–based Robust Design Optimization. Robust objectives and constraints can be formulated by means of worst–case bounds on two risk metrics, i.e. quantile and tail conditional expectation of the output distributions. Such bounds can then be refined on the basis of partial knowledge available for the specific problem at hand or by adding higher moment information. The required output moments are estimated by two related reduced quadrature techniques, which use the first four moments of the input distributions. As demonstrated by means of an aircraft sizing test case, the methodology is of immediate implementation and deployment, and features a convenient trade–off in cost and accuracy. It is therefore thought to be very promising for industrial applications.
2. Keywords: Robust Design Optimization, Chebyshev Inequalities.

3. Introduction
Very promising results can be found, in principle, by coupling design analysis tools with optimization algorithms. However, designs of practical interest can be critically sensitive to several design variables and parameters, the value of which could change unpredictably during subsequent stages of the design and production process, or in flight. Such sensitivity may be exacerbated by optimization practices. In fact, the nominal performance of the optimum design could be significantly degraded by the mentioned unforeseen variations. Furthermore, if optimality is found on the design constraint boundaries, such variations may render the optimal design solution unfeasible. More articulated approaches, such as Robust Design Optimization (RDO), are hence required. The practical hurdles of RDO are, however, manifold. On the one hand, the problem has to be formulated to find a correspondence between high level goals – such as the guarantee of performance and the containment of risks – and their mathematical representation. On the other hand, such formulation has to be implementable in an efficient yet sufficiently accurate way. To tackle such issues, the research reported in [11] proposed a novel interpretation of typical mean-variance formulations for robust design optimization. Robust objectives and constraints were cast either by quantile or tail conditional expectation (TCE) metrics. Such metrics were approximated by bounds built by weighted sums of mean and standard deviation of the function of interest. The weights were chosen depending on the required probability level and could account for some form of partial knowledge on the considered distributions (e.g. symmetry of the output density function). The required moments were calculated through the Univariate Reduced Quadrature (URQ) [10]. The present paper expands upon that work along the following directions:

1. Provide further distributional assumptions as modelling options for objectives and constraints;
2. Account for the case in which also skewness and kurtosis of the output functions are available. Obtaining such moments calls for the use of a higher order propagation method. In the present paper, we adopt for this purpose a Bivariate Reduced Quadrature technique (BRQ) [3, 10];
3. Extend the use of probabilistic inequalities to the input variables. This has a double advantage: on the one hand, it allows to specify the moments required by URQ and BRQ without deducing them from input distributions, which might be not known. On the other hand, it supplies a practical, distributional–free formulation for the probabilistic box constraints on the input variables.

The paper is structured as follows. Section 4 defines a general Design Optimization problem under uncertainty, and introduces quantile and TCE metrics. Section 5 translates such problem into a MinMax formulation, which enables to approximate quantile and TCE metrics as a function of a given number of output moments. Such moments can be conveniently estimated by the reduced quadrature techniques that are summarized in Section 6. Section 7 describes the application of the proposed methodology to a test case of industrial relevance and finally Section 8 presents conclusions and future work.

4. General Problem Formulation

Assume that the design analyses are performed by the (generally nonlinear) functions $f(\boldsymbol{x})$ and $g_i(\boldsymbol{x})$, $i = 1, 2, \dots I$, where $\boldsymbol{x} \in \mathrm{R}^n$ is the vector of design variables. The *deterministic design optimization problem* is hence formulated as follows:

$$\begin{aligned} &\min_{\boldsymbol{x}\in\mathbb{R}^n} f(\boldsymbol{x}), \\ &\text{s. t.: } g_i(\boldsymbol{x}) \leqslant 0, i = 1, 2, \dots, I, \\ &\text{and: } \boldsymbol{x}_L \leqslant \boldsymbol{x} \leqslant \boldsymbol{x}_U. \end{aligned} \tag{1}$$

The *design optimization problem under uncertainty* resulting from the stochastic modeling of the input variables (an analogous reasoning could be made with respect to uncertain parameters) can be written as follows:

$$\begin{aligned} &\min_{\mu_{\boldsymbol{x}}\in\mathbb{R}^n} \theta\big[f(\boldsymbol{x},\ P_{0f})\big], \\ &\text{s. t.: } \zeta_i\big[g_i(\boldsymbol{x}),\ P_{0g_i}\big] \leqslant 0,\ i = 1, 2, \dots, I, \\ &\text{and: } \chi\big[\boldsymbol{x},\ \boldsymbol{x}_L,\ \boldsymbol{x}_U,\ P_{\boldsymbol{x}}\big] \leqslant 0, \end{aligned} \tag{2}$$

where:

- $\boldsymbol{x}$ is a vector of random variables with mean $\mu_{\boldsymbol{x}}$;
- θ is a metric that models the desirability of a given output behaviour, expressed as function of a given probability level P_{0f}, and is a function of the probability density function (PDF) of the deterministic objective p_f induced by the multivariate input probability distribution function $p_{\boldsymbol{x}}$;
- ζ_i and χ are metrics that quantify the probabilistic satisfaction of the ith deterministic constraint and the input bounds, with tolerance levels P_{0g_i} and $P_{\boldsymbol{x}}$, respectively.

θ, ζ and χ are the mathematical expressions of the rationale behind the design approach to coping with uncertainty. The significance and interpretability of the optimal design solutions depends on how well such rationale is embodied in the problem. For this reason, we propose to formulate θ, ζ and χ through risk metrics that can be of direct use to engineering decision making. Two of such metrics, which we define *quantile* and *Tail Conditional Expectation* (TCE), are closely related to those that have been popularized in the financial field under the names of *Value at Risk* (VaR) and *Conditional VaR* (CVaR), respectively [15]. They embody two different kinds of risk perception. Consider for sake of simplicity the continuous function $y(\boldsymbol{x})$, which can represent either f or g_i. The P_0-quantile of its distribution, defined as the value y_{P_0} such that $P(y(\boldsymbol{x}) \leqslant y_{P_0}) = P_0$, represents the worst specimen over a given percentage (P_0) of acceptable designs. Hence it primarily focuses on the *limit case* which has to be avoided in an absolute sense (e.g. an ultimate load). In many design cases, however, it is necessary to handle cases in which the quantile is exceeded, and it does matter to quantify how much; a typical example is the design of fail–safe systems. The TCE, which weighs the unacceptable designs in y by their distance from a target quantile can be useful in such cases. It has the following mathematical definition:

$$\text{TCE}_{1-P_0}(y) = E\left[y|y \geqslant y_{P_0}\right] = \frac{1}{1-P_0}\int_{\Omega} y(\boldsymbol{x})p_{\boldsymbol{x}}(\boldsymbol{x})d\boldsymbol{x}, \tag{3}$$

where $\Omega = \{\boldsymbol{x}\ :\ y(\boldsymbol{x}) \geqslant y_{P_0}\}$ is the region of unacceptable designs, i.e. those exceeding the P_0-quantile.

4. MinMax Formulation

Calculating the quantile or TCE of the output distributions at each step of the optimization is often computationally intractable for practical cases of engineering interest. For this reason, affordable yet meaningful approximations of quantile and TCE metrics are sought. By noticing that the estimation of the first output moments can be much cheaper than the estimation of quantiles and TCE, we focus on bounds that result from the generalization of the classical Chebyshev inequality: $P(|y(\boldsymbol{x}) - \mu_y|) \geqslant k_y\sigma_y) \leqslant \frac{1}{k_y^2}$, where k_y is a real number, $\mu_y < \infty$ and $\sigma_y^2 \neq 0$ are the mean and variance of y. In particular, extensions to one–sided inequalities [1] and to conditional expectations [9] allow to calculate quantile and TCE bounds, respectively, as functions of mean and variance. Such bounds may then be further refined by including available partial knowledge about the function at hand. Call ρ the considered risk metric (either the quantile or the TCE). Several computational procedures (see for example [13, 17, 6, 20]) have been proposed to cast the process of finding such bounds as an optimization that supplies the worst–case metric $WC\rho$ for the random variable y in the class of distributions $\mathcal{Y}$ defined by given moments $m_{y,j}$ (i.e. $m_{y,1} = \mu_y$, $m_{y,2} = \sigma_y^2$, etc.) and associated assumption $\mathcal{A}$ (e.g. bounded support, unimodality, symmetry, etc.):

$$WC\rho(y) = \max_{\substack{y \in \mathcal{Y}(m_{y,j})_{\mathcal{A}} \\ j=1,\dots,J}} \big(\rho(y)\big), \tag{4}$$

By suitably inserting specialized forms of the $WC\rho(y)$ into Eq. 2, the following MinMax problem is obtained:

$$\begin{aligned}
&\min_{\mu_{\boldsymbol{x}} \in \mathbb{R}^n} \left(\max_{\substack{f \in \mathcal{F}(m_{f,j})_{\mathcal{A}_f} \\ j=1,\dots,J}} \rho\Big(f(\boldsymbol{x})\Big) \right), \\
&\text{s. t.: } \left(\max_{\substack{g_i \in \mathcal{G}_i(m_{g_i,j})_{\mathcal{A}_{g_i}} \\ j=1,\dots,J}} \rho\Big(g_i(\boldsymbol{x})\Big) \right) \leqslant 0, \quad i = 1, 2, \dots, I, \\
&\text{and: } \begin{cases} \Big(\max_{\substack{x_k \in \mathcal{X}_k(m_{x_k,s})_{\mathcal{A}_{x_k}} \\ s=1,\dots,S}} \rho\big(x_k\big)\Big) \leqslant x_{U_k} \\ \Big(\max_{\substack{x_k \in \mathcal{X}_k(m_{x_k,s})_{\mathcal{A}_{x_k}} \\ s=1,\dots,S}} \rho\big(-x_k\big)\Big) \geqslant x_{L_k} \end{cases} \quad k = 1, \dots, n.
\end{aligned} \tag{5}$$

This is equivalent to solving Problem 2 as a double optimization loop. In the inner loop, which has to be solved at each step of the outer optimization process, worst-case approximations are found for the metrics of interest, as a function of a given number of estimated output moments, under suitable distributional constraints. In the outer loop, such metrics are minimized, subject to worst–case constraints and input bounds. In this paper, we turn to a considerably simpler case, in which simplified or closed–form expressions for the inner optima are available. In particular, we collect existing results for which $WC\rho(y)$ takes the explicit form $WC\rho(y) = \mu_y + k_y\sigma_y$, where k_y is a function of a given probability and optional additional information. This yields the following simplified formulation for Problem 5:

$$\begin{aligned}
&\min_{\mu_{\boldsymbol{x}} \in \mathbb{R}^n} \theta = \mu_f + k_f \sigma_f, \\
&\text{s. t.: } \zeta_i = \mu_{g_i} + k_{g_i}\sigma_{g_i} \leqslant 0, i = 1, 2, \dots, I, \\
&\text{and: } \boldsymbol{x}_L + k_{\boldsymbol{x}_L}\sigma_{\boldsymbol{x}} \leqslant \mu_{\boldsymbol{x}} \leqslant \boldsymbol{x}_U - k_{\boldsymbol{x}_U}\sigma_{\boldsymbol{x}}.
\end{aligned} \tag{6}$$

Table 1 summarizes the value of the weights for the cases of quantile and TCE bounds on distributions for which the first two moments are known. For comparison purposes we also include the value of the corresponding weights for the exact quantile and TCE of a Normal distribution with given mean and variance. Table 2 summarizes the value of the weights for quantile and TCE bounds for cases in which the first four moments are known and no additional assumption is imposed. Please note that, in the general case, the bounds in Table 2 are implicit, but have straightforward numerical solution. By comparing quantile and TCE bounds in Tables 1 and 2, we can notice that, in most of the cases shown, there are probability ranges for which $k_y(P_0)$ has the same expression for the two metrics. Since $TCE_{1-P_0}(y) \geqslant y_{P_0}$ by definition, it holds that: $y_{P_0} \leqslant TCE_{1-P_0} \leqslant \mu_y + k_y\sigma_y$. Therefore, the TCE bound is in those cases less conservative than the corresponding quantile bound. The analogy existing between Problem 6 and classical mean–variance RDO formulations is striking (see for example [18, 12]). However, Problem 6 incorporates a much richer theoretical framework, which has significant practical consequences, including but not limited to the following ones:

- objectives and constraints can be modelled by means of risk metrics such as quantile or TCE, which can be chosen by the designer depending on the problem at hand;
- k_f is a well defined parameter that depends on the metric of choice, on the desired probability level and that can be refined by using qualitative knowledge on f in the uncertain parameters domain.

Table 1: Value of k_y for quantile and TCE upper bounds with known μ_y, σ_y

<table>
<tr><th>Validity</th><th>Quantile</th><th>TCE</th></tr>
<tr><td>No distributional assumption</td><td>[1]</td><td>[9]</td></tr>
<tr><td>$0 \leq P_0 \leq 1$</td><td colspan="2">$k_y = \sqrt{\frac{P_0}{1-P_0}}$</td></tr>
<tr><td>Bounded support: $y \in [A,\, B]$</td><td colspan="2">[5]</td></tr>
<tr><td>$0 \leq P_0 \leq \frac{1}{1+k_A^2}$</td><td>$k_y = \frac{k_A(k_A+k_B)P_0}{(k_A+k_B)(1-P_0)-k_A}$</td><td>$k_y = \frac{k_A P_0}{(1-P_0)}$</td></tr>
<tr><td>$\frac{1}{1+k_A^2} \leq P_0 \leq \frac{k_B^2}{1+k_B^2}$</td><td colspan="2">$k_y = \sqrt{\frac{P_0}{1-P_0}}$</td></tr>
<tr><td>$P_0 \geq \frac{k_B^2}{1+k_B^2}$</td><td colspan="2">$k_y = k_B$</td></tr>
<tr><td></td><td colspan="2">with: $k_A = \frac{\mu_y - A}{\sigma_y}$ and $k_B = \frac{B-\mu_y}{\sigma_y}$, with $k_A k_B \geqslant 1$</td></tr>
<tr><td>Symmetry</td><td>[13]</td><td>[2]</td></tr>
<tr><td>$0 < P_0 \leq \frac{1}{2}$</td><td>$k_y = 0$</td><td>$k_y = \frac{\sqrt{P_0}}{\sqrt{2}(1-P_0)}$</td></tr>
<tr><td>$\frac{1}{2} < P_0 < 1$</td><td colspan="2">$k_y = \frac{1}{\sqrt{2(1-P_0)}}$</td></tr>
<tr><td>Unimodality</td><td>[21]</td><td rowspan="3">Not available</td></tr>
<tr><td>$0 \leq P_0 \leq \frac{5}{6}$</td><td>$k_y = \sqrt{\frac{3P_0}{(3P_0-2)}}$</td></tr>
<tr><td>$\frac{5}{6} < P_0 < 1$</td><td>$k_y = \sqrt{\frac{9P_0-5}{9(1-P_0)}}$</td></tr>
<tr><td>Symmetry and Unimodality</td><td>[13]</td><td>[22]</td></tr>
<tr><td>$0 < P_0 \leq \frac{1}{3}$</td><td rowspan="2">$k_y = 0$</td><td>$k_y = \frac{2\sqrt{P_0}}{3(1-P_0)}$</td></tr>
<tr><td>$\frac{1}{3} \leq P_0 \leq \frac{1}{2}$</td><td rowspan="2">$k_y = \sqrt{3}P_0$</td></tr>
<tr><td>$\frac{1}{2} \leq P_0 \leq \frac{2}{3}$</td><td rowspan="2">$k_y = \sqrt{\frac{2}{9(1-P_0)}}$</td></tr>
<tr><td>$\frac{2}{3} \leq P_0 < 1$</td><td>$k_y = \frac{2}{3\sqrt{(1-P_0)}}$</td></tr>
<tr><td>Normality (exact value)</td><td></td><td>[8]</td></tr>
<tr><td>$0 \leq P_0 \leq 1$</td><td>$k_y = \Phi^{-1}(P_0)$</td><td>$k_y = \frac{\phi(\Phi^{-1}(P_0))}{1-P_0}$</td></tr>
<tr><td></td><td colspan="2">where ϕ is the PDF of the standard Gaussian variable and Φ is its cumulative distribution function</td></tr>
</table>

This can be useful in many cases of practical interest, for which such knowledge might be derived from physical considerations, but also lower order theories, surrogate models and so on;

- k_{g_i} is not uniquely based on a normality assumption, as typically done in RDO, but can account for more realistic cases, by including qualitative knowledge on the underlying distributions, as for the objective;

- The relaxation of distributional assumptions can also be useful to model input bounds. This turns out to be very useful towards a completely distributional–free RDO methodology, to tackle cases in which the only available information about the input distributions consists in a certain number of moments.

Table 2: Value of k_y for quantile and TCE upper bounds with known μ_y, σ_y, γ_y and Γ_y

Validity	Quantile	TCE
No distributional assumption	[5]	
$0 < P_0 \leq \frac{1}{2}\left(1 + \frac{\gamma_y}{\sqrt{\gamma_y^2+4}}\right)$	$W\Big(V(k_y)\Big) = P_0$	$k_y = -\frac{P_0}{1-P_0} V(k_y)$
$\frac{1}{2}\left(1 + \frac{\gamma_y}{\sqrt{\gamma_y^2+4}}\right) < P_0 < 1$	$W(k_y) = 1 - P_0$	
	with: $R(u) = \gamma Q(u) + \Delta(u), \quad S(u) = Q(u) + \Delta(u)$; $Q(u) = 1 + \gamma u - u^2, \quad \Delta = \Gamma - \gamma^2 - 1$; $W = \frac{\Delta}{Q^2(u)+\Delta(1+u^2)}, \; V = \frac{R(u) - \sqrt{R^2(u)+4Q(u)S(u)}}{2Q(u)}$	

Finally, we show for completeness in Table 3 the value of $k_y(P_0)$ for the cases of two-tailed probability bounds for known mean and variance. Such bounds can be especially useful for "nominal–the–best" variants of Problem 5, in which ρ represents the absolute deviation from the mean.

Table 3: Value of k_y for two-tailed probability upper bounds for known μ_y, σ_y

Validity	
No distributional assumption	[14]
$0 \leq P_0 \leq 1$	$k_y = \frac{1}{\sqrt{1-P_0}}$
Unimodality	[14]
$0 \leq P_0 \leq \frac{5}{6}$	$k_y = \frac{2}{\sqrt{(4-3P_0)}}$
$\frac{5}{6} < P_0 < 1$	$k_y = \frac{2}{3\sqrt{(1-P_0)}}$
Symmetry and Unimodality	[16]
$0 \leq P_0 \leq \frac{2}{3}$	$k_y = \sqrt{3}P_0$
$\frac{2}{3} \leq P_0 < 1$	$k_y = \frac{2}{3\sqrt{(1-P_0)}}$
Normality (exact value)	
$0 \leq P_0 \leq 1$	$k_y = \Phi^{-1}(\frac{1+P_0}{2})$

6. Quantification and Propagation of Input Uncertainty

The quantification of input uncertainties is usually performed prior to the design optimization process. The mathematical model is chosen to match as closely as possible the uncertain knowledge about the design problem being considered, which is either based on statistical data or relies on expert opinion, or a combination of both. In this paper, we consider the case in which such model is expressed in terms of the first four moments of the input marginal distributions, and we denote by $\gamma_p = E(x_p - \mu_{x_p})^3/\sigma_{x_p}^3$ and $\Gamma_p = E(x_p - \mu_{x_p})^4/\sigma_{x_p}^4$ the skewness and the kurtosis of the p^{th} input variable, respectively. In the case of bounded input distributions ($x_p \in [A_p,\ B_p]$, with $A_p,\ B_p \in \mathbb{R}$), by writing $k_{A_p} = (\mu_{x_p} - A_p)/\sigma_{x_p}$ and $k_{B_p} = (B_p - \mu_{x_p})/\sigma_{x_p}$, the four moments are regulated by the following inequalities [6]:

$$\begin{aligned} -k_{A_p} &\leqslant 0 \leqslant k_{B_p} \\ 1 &\leqslant k_{A_p} k_{B_p} \\ \frac{1-k_{A_p}^2}{k_{A_p}} &\leqslant \gamma_{x_p} \leqslant \frac{k_{B_p}^2 - 1}{k_{B_p}} \\ 0 \leqslant \Gamma_{x_p} - \gamma_{x_p}^2 - 1 &\leqslant \frac{(k_{B_p}\gamma_{x_p} - k_{B_p}^2 + 1)(k_{A_p}\gamma_{x_p} + k_{A_p}^2 - 1)}{1 - k_{A_p}k_{B_p}}, \end{aligned} \tag{7}$$

where the last inequality holds if and only if $1 < k_{A_p}k_{B_p}$. Please note that the last two inequalities can be sharpened by considering unimodality of the input marginals [19]. Knowledge of such bounds can be useful to parametrically set up the input uncertainty models without referring explicitly to input distributions. In addition, the optimization bounds can also be set up in the form given in Problem 6 without complete knowledge of the input marginal distributions, by using the functional relationships in Table 2. For the propagation phase, we consider two methods that use such moments to obtain either the first two moments of the output distributions (Univariate Reduced Quadrature [10]) or the first four moments (Evans' method [3], that we call Bivariate Reduced Quadrature for consistency). For the sake of brevity, we refer the reader to [10, 3] for a detailed explanation of the propagation methods. In this paper it is sufficient to underline that in both cases the propagation is carried out by suitably weighting a number of functional outputs corresponding to selected input values, which are for the BRQ the following $2n^2 + 1$ sampling points:

$$\boldsymbol{x}_{p\pm} = \mu_{\boldsymbol{x}} + h_p^{\pm}\sigma_{x_p}\boldsymbol{e}_p, \tag{8}$$

$$\boldsymbol{x}_{p\pm q\pm} = \mu_{\boldsymbol{x}} + h_p^{\pm}\sigma_{x_p}\boldsymbol{e}_p + h_q^{\pm}\sigma_{x_q}\boldsymbol{e}_q, \tag{9}$$

where $\boldsymbol{e}_p$ is the p^{th} vector of the identity matrix of size n and $h_p^{\pm}$ are given by:

$$h_p^{\pm} = \frac{\gamma_{x_p}}{2} \pm \sqrt{\Gamma_{x_p} - \frac{3\gamma_{x_p}^2}{4}}. \tag{10}$$

On the other hand, the URQ requires only the $2n+1$ sampling points $\boldsymbol{x}_{p\pm}$ in Eq. 8.

7. Robust Aircraft Design

A possible objective of robust approaches at the conceptual design stage is to render the optimal design insensitive to variations that are likely to occur downstream in the development process, with the purpose of avoiding nugatory iterations between design phases. Numerically, this can be achieved by introducing uncertainty in the decision variables. To illustrate such procedure, we consider the aircraft sizing test case that was used for the European Union Integrated Project VIVACE [4]. The deterministic optimal sizing can be formulated as in Problem 1. The objective is the Maximum Take-Off Weight: $f = MTOW(\boldsymbol{x})$, which has to be minimized with respect to the design variables $\boldsymbol{x}$, subject to the following constraints:

1. Approach speed: $v_{app} < 120$ Kt $\Rightarrow g_1 = v_{app} - 120$;
2. Take-off field length: $TOFL < 2000$ m $\Rightarrow g_2 = TOFL - 2000$;
3. Percentage of total fuel in wing tanks: $K_F > 0.75 \Rightarrow g_3 = 0.75 - K_F$;
4. Percentage of sea-level thrust available in cruise: $K_T < 1 \Rightarrow g_4 = K_T - 1$;
5. Climb speed: $v_{zclimb} > 500$ ft/min $\Rightarrow g_5 = 500 - v_{zclimb}$;
6. Range: $RA > 5800$ Km $\Rightarrow g_6 = 5800 - RA$.

Table 4: Considered design variables.

Design variables	Definition [units]	Det. Opt. Bounds $[\boldsymbol{x}_L,\ \boldsymbol{x}_U]$	Uncertain modelling $\sigma_{\boldsymbol{x}}$	$\gamma_{\boldsymbol{x}}$	$\Gamma_{\boldsymbol{x}}$
$x_1 = S$	Wing area [m^2]	[140, 180]	2	-0.5	1.7
$x_2 = BPR$	Engine bypass ratio [ ]	[5, 9]	0.5	0.5	2.5
$x_3 = b$	Wing span [m]	[30, 40]	0.2	-0.3	3
$x_4 = \Lambda$	Wing sweep [deg]	[20, 30]	1.5	0	3
$x_5 = t/c$	Wing thickness to chord ratio [%]	[7, 12]	0.005	-0.2	2.0
$x_6 = T_{e_{SL}}$	Engine sea level thrust [kN]	[100, 150]	10	0.5	2.4
$x_7 = FW$	Fuel weight [Kg]	[12000, 20000]	500	-0.5	2.3

Table 4 provides descriptions of the design variables with their permitted ranges. The considered case is a twin–jet civil aircraft with 150 passengers. Cruise Mach and altitude are 0.75 and 31000 ft, respectively. We model the uncertainty in the design variables by reasoning on their moments, which are also shown in Table 4. The underlying random variables are supposed not to change in dispersion and shape within the design space. In particular, the skewness is defined to model asymmetries, while the kurtosis is interpreted as a measure of the density center/tails ratio. Please note that the value of the kurtosis and skewness that we have chosen for the wing area is such that its distribution is not unimodal ($\Gamma_S - \gamma_S^2 = 1.45 \leqslant 1.52$ [7]). This is an artifice that is useful to explain the properties of the proposed methods when working with arbitrary distributional shapes. We perform three sets of numerical experiments:

DDO The Deterministic Design Optimization, which supplies a baseline for the robust optimization;

RC2 A robust optimization in which objectives and constraints are modeled through TCE bounds based on the first two moments, obtained by the URQ;

RC4 A robust optimization that uses the first four moments, obtained by the BRQ, to model constraints and objectives as TCE bounds.

The optimization problems are solved by using Matlab's gradient-based constrained optimizer `fmincon`, version R2010b. This algorithm uses a sequential quadratic programming method, in which a quadratic programming subproblem is solved at each iteration. A quasi-Newton estimate of the Hessian of the Lagrangian is obtained by using the Broyden–Fletcher–Goldfarb–Shanno formula. The results of the deterministic optimization are shown in Table 5. The fact that most of the constraints are active is

Table 5: Solution of DDO that serves as starting point for RC2 and RC4.

Input variables		Obj./Constr.	
S [m^2]	144.90	f [Kg]	77007
BPR [ ]	9.00	g_1 [Kt]	-2.40
b [m]	40.00	g_2 [m]	-13.90
Λ [deg]	20.00	g_3 [ ]	0.00
t/c [%]	8.3	g_4 [ ]	0.00
$T_{e_{SL}}$ [kN]	101.60	g_5 [ft/min]	0.00
FW [Kg]	14817	g_6 [Km]	0.00

symptomatic of the lack of robustness of this solution to the slightest input variation. To tackle such issue we turn to Problem RC2, which can be formulated as Problem 6, and we seek suitable metrics for the objectives and constraints. The input variables box constraints are modelled through quantile bounds given in Table 2, with a probability of exceeding the upper or lower tail bounds of 0.2 for each of the variables. Please note that resorting to upper bounds requires lower tail quantiles to be modelled by taking the opposite of the considered distributions, and hence the opposite of the uneven moments. Since dispersion and shape parameters are assumed not to vary during the optimization, this choice only requires solving the implicit bounds equation once for each of the variables. This can be done before the beginning of the optimization and results in shrinking the design space. The following step consists in choosing a suitable assumption for the robust objective and constraints. In a first attempt, suppose that none of the available assumptions is applicable – this may be due for example to a possible multimodality of output distributions induced by the chosen input distributions. We could therefore model objective and constraints through the generic TCE metric in Table 1, by taking $P_0 = 0.9$ for the objective and the constraints. This leads to the RC2 solution shown in Table 6. The optimal design is heavier than the original deterministic design of about 4800 Kg. In fact, we have traded performance for feasibility robustness, as we can see by post–processing the optimal results by means of a Monte Carlo simulation. We draw $2 \cdot 10^5$ samples from a set of Pearsons' marginal distributions with the considered input moments. In this particular case (an infinite set of distributions exist with the given moments), the constraints are satisfied with a probability that is higher than 99%. Using worst–case bounds for the objective and the constraints causes such result to be more conservative than what we asked for; for example, in the case of the objective, the TCE calculated on the sample is approximately 1.2% lower than the TCE estimated by worst–case bounds: $TCE_{0.1,\,MCS}(MTOW) = 82803$ Kg $< TCE_{0.1,\,RC2}(MTOW) = 83813$ Kg. There are several possible approaches that can be implemented in the proposed framework to alleviate such overconservativeness. In the first place, we could refine the adopted distributional assumptions, for example by bounding the output quantities (and their respective distributions) by means of physical considerations. Alternatively, we could adopt an uncertainty management strategy that cuts the output distribution to given maximal quantities, which might be defined by the chief engineer. Anything below the given threshold will have to be covered by robustness, i.e. will have to be absorbed passively by the design. Anything above that level represents a risk that has to be covered by the project financially, since it might require major design fixes, or by alternative technical options. To numerically appreciate such a strategy, we might further develop on the obtained robust solution, which has minimized the worst–case $TCE_{0.1}(MTOW)$. By building on the fact that the $f = MTOW$ has a positive and bounded distribution (with an unknown upper bound B), we determine from Eq. 7 that $B \geqslant (\mu_f^2 + \sigma_f^2)/\mu_f$. By introducing the inequality in the expression of the worst–case TCE in Table 1 for the case of bounded support, we deduce the following equation [5]:

$$\max_{\substack{f \in \mathcal{F}(\mu_f,\sigma_f^2)_{\mathcal{A}} \\ \mathcal{A}: f \in [0,\,(\mu_f^2+\sigma_f^2)/\mu_f]}} TCE_{1-P_0}(f) = \begin{cases} \frac{\mu_f^2+\sigma_f^2}{\mu_f} & P_0 \geqslant \frac{\sigma_f^2}{\mu_f^2+\sigma_f^2} \\ \\ \frac{\mu_f}{1-P_0} & P_0 \leqslant \frac{\sigma_f^2}{\mu_f^2+\sigma_f^2}. \end{cases} \tag{11}$$

Such result offers the decision maker the opportunity to pick within a given range the desired protection level against the modelled uncertainties. In the numerical example, this range is [81917, 83813] Kg. We can verify that the Monte Carlo estimate of the TCE given above lies within the interval. Please note that such an approach can be also applied to the constraints. Finally, the third numerical experiment that we have carried out deals with yet another way of reducing the overconservativeness of the worst–case approach. By increasing the number of estimated output moments, more refined bounds can be built on the output metrics of interest. We have therefore chosen to model objectives and constraints through TCE bounds, with the same probability level used for RC2. This leads to the RC4 solution that is also shown in Table 6. As expected, the robust design for RC4 is lighter than for RC2,

Table 6: Solutions of RC2 and RC4 Problems.

Input variables	RC2	RC4	Nominal features	RC2	RC4	Robust features	RC2	RC4
S [m^2]	155.70	152.53	f [Kg]	81914	80624	θ [Kg]	83813	81696
BPR [ ]	8.13	8.13	g_1 [Kt]	−1.21	−1.49	ζ_1 [Kt]	0.00	−0.00
b [m]	39.69	39.69	g_2 [m]	−161.02	−126.91	ζ_2 [m]	−43.77	−46.89
Λ [deg]	22.43	22.43	g_3 [ ]	−0.24	−0.14	ζ_3 [ ]	−0.00	0.00
t/c [%]	10	9.3	g_4 [ ]	−0.14	−0.13	ζ_4 [ ]	−0.11	−0.11
$T_{e_{SL}}$ [kN]	110.62	107.43	g_5 [ft/min]	−97.45	−61.97	ζ_5 [ft/min]	−0.00	−0.06
FW [Kg]	17251	16553	g_6 [Km]	−543.05	−377.58	ζ_6 [Km]	0.00	0.00

and still it respects the given constraints. This is mainly due to a better accuracy in modelling the constraints to the required probability of feasibility, which allows the optimizer to explore areas of the design space that were considered unfeasible in RC2. The same happens with the objective function:

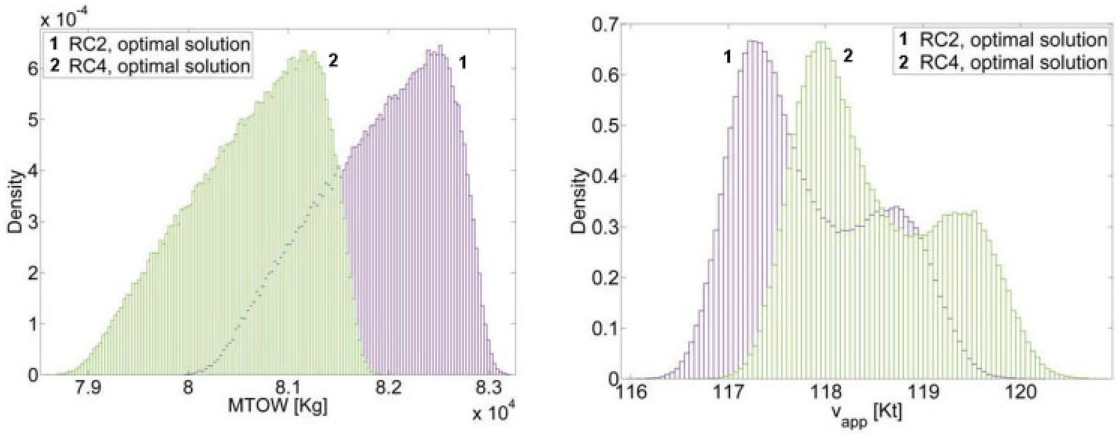

Figure 1: Comparative results of two Monte Carlo analyses with $2 \cdot 10^5$ samples on RC2 and RC4 solutions.

$TCE_{0.1,\,MCS}(MTOW) = 81512$ Kg $< TCE_{0.1,\,RC4}(MTOW) = 81696$ Kg, which reduces the bounds conservativeness to the 0.2%. To appreciate the differences between RC2 and RC4 results, the probability density functions of the objective and of one of the constraint are shown in Figure 1. It can also be noticed that the v_{app} distribution is actually bimodal for the chosen input variables modelling. Nevertheless, the improved accuracy has an impact on the computational cost: deploying the BRQ to calculate the first four moments requires for the presented application 99 function evaluations (and the solution of the implicit equations in Table 2) for each evaluation of the robust objective or constraint during the optimization process, instead of 15 evaluations required by the URQ. This may suggest the deployement of adaptive optimization strategies that would use the URQ in the preliminary phases of the optimization, and turn to the BRQ in the neighbourhood of the optimum solution, or in presence of strong non-linearities in the functional outputs. The fact that the URQ sampling points are a subset of the BRQ sampling points may increase the effectiveness of such strategies.

8. Conclusions and Future Work

The paper reports a comprehensive methodology to cast and deploy moment–based robust design optimization. The objectives and constraints are coinceived as risk metrics that are of direct interest to the designer, namely quantile and tail conditional expectation. Such metrics are efficiently accounted for in the numerical methodology by resorting to upper bounds, which are function of a limited number of moments and can be refined by using partial knowledge about the problem at hand. In the general case, such bounds are computed by optimization techniques. This yields a MinMax framework that embeds the search for the worst–case bounds within the robust optimization algorithm. For many cases of practical interest, which are collected in the paper in the form of tables, such framework reduces to a simpler formulation that resembles classical RDO mean–variance formulations. However, in contrast to classical RDO, the proposed framework supplies guidelines for the choice of the weight functions, which enables to consistently exploit higher moments or partial distributional information, and supplies a rationale to interpret the results. The required moments are obtained in this work by using either a Univariate Reduced Quadrature or its Bivariate counterpart. By means of an aircraft sizing test case it has been shown that the methodology enables the designer to reason about the suitable robustness level to be accounted for in the design. This can be very useful for industrial applications. Besides, the use of several modelling options for objectives and constraints and of the two mentioned propagation techniques has allowed to compare the accuracy and the computational cost of several robust solutions obtainable through the proposed framework. Future work will focus on the development of adaptive strategies, and the extension of the proposed methodology to account for joint constraints and multi–objective problems.

9. References

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