Ensemble of Unified Reliability Formulations (EURF)

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1. Abstract
Various Reliability-Based Design Optimization (RBDO) methods have been developed and widely used to solve design optimization problems with the existence of design uncertainties. The general problem formulation states that the objective function is minimized while the failure probabilities of the performance constraints are subjected to the allowable probability levels. RBDO algorithms derive and formulate various approximate probabilistic constraints with respect to the means of the randomly distributed design variables in replace of the evaluations of failure probabilities using the integral of joint probability density functions. However, there is a huge diversity of approximate probabilistic formulations from various RBDO algorithms. The goodness of fit of each approximate model is problem dependent but highly affects the accuracy and efficiency of the optimization process. In this paper, a Unified Reliability Formulation (URF) is derived from the fundamental aspect of the linear expansion with allowable reliability level to provide a general category of first-order RBDO methods. The URF is determined by the linear expansion at an Allowable Reliability Point (ARP) with the sensitivity analysis associated with a Gradient-based Transformation Point (GTP). The reliability of the ARP is exactly equal to the allowable probability. The GTP is the chosen expansion point of the approximate probabilistic constraint in each RBDO algorithm. The derived URF not only provides a comprehensive understanding of approximate probabilistic constraint but also an insightful acknowledgment of how various RBDO algorithms can be unified into one general equation. For instance, the various formats of the URFs for the existing RBDO algorithms are demonstrated. The accuracy of each URF depends on the evaluations at the GTPs. Therefore, an Ensemble of Unified Reliability Formulations (EURF) is formed to group together the approximate probabilistic formulations from various RBDO algorithms. The intersection of the URFs from each RBDO algorithm is considered when the limit state is a convex function; on the other hand, the union of the URFs is considered for the concave limit state function. EURF covers a wider range of reliability analyses than any individual method. The benchmark examples show that the EURF requires fewer iteration to finds the optimal solutions than either RIA or PMA when dealing with highly nonlinear constraints.
2. Keywords: ensemble of probabilistic constraints; Unified Reliability Formulation (URF); reliability analysis; Allowable Reliability Point (ARP); Gradient-based Transformation Point (GTP); Chance Constrained Programming (CCP); Reliability Index Approach (RIA); Performance Measure Approach (PMA).

3. Introduction
The deterministic design optimization problems, which minimize the objective (or cost) functions subject to the performance constraints, have been utilized to find the optimal engineering designs satisfying the optimality and feasibility conditions simultaneously. However, the deterministic optimal solutions, in fact, have high probabilities (around 50% or even more) of system failures because of the existence of the design uncertainties. To decrease the failure probability and satisfy the acceptable reliability level, the more conservative design variables are desired on the compromise on the optimality. The Reliability-Based Design Optimization (RBDO) problems have been formulated to minimize the cost function while the failure probabilities of the performance constraints are subjected to the allowable levels. Nevertheless, the evaluations of the failure probabilities, which require the multivariate integrations associated with the joint probability density functions (JPDF), are computationally costly. Many RBDO algorithms have been developed to transform the probabilistic constraints into solvable deterministic formulations and they have been widely utilized to solve the optimization problems with design uncertainties.
In RBDO, Cornell [1] first defined a reliability index as the ratio of the negative expected value of the performance constraint function over its standard deviation in order to represent the reliability level of the design. Accordingly, the probability of the system failure can be evaluated by the standard normal cumulative distribution function (CDF) of the negative value of the reliability index [2]. Following an inverse transformation of the standard normal CDF [3], the probabilistic performance constraint can be analytically transformed to a deterministic formulation, where the reliability index of the design is subjected to a minimum level. During the optimization process, the design point, which violates the constraint of the reliability index,

will have failure probability higher than the acceptable level; contrarily, the one, which satisfies the reliability-based constraint, will fulfill the requirement of the reliability level. Using the mean-value first-order approximation of second moment to evaluate the Cornell reliability index, the Chance Constrained Programming (CCP) [4, 5] has been developed. In the CCP, the probabilistic constraints are linearly approximated associated with the mean values of the random design variables and they are iteratively updated with respect to the new design points. However, the mean-value analysis of the reliability index is inaccurate when evaluating the failure probabilities associated with the nonlinear constraints [2].

In order to improve the accuracy of the reliability analysis, Veneziano [6] collected an approximate set of reliability indices along the tail of the random distribution and evaluated the failure probability by the smallest one in the set. Hasofer and Lind [7] further found the minimum reliability index at the foot of the perpendicular from the origin to the tail approximation of the constraint in the standard normal space, that is, the normal tail approximation [8]. The foot point, also known as the Most Probable Failure Point (MPFP) [7, 9, 10], is crucial for the evaluation of the failure probability and it is determined by finding the shortest distance from the origin to the limit state of the performance constraint measured in the standard normal space. The Reliability Index Approach (RIA) method [2, 11-17], which utilizes the First Order Reliability Method (FORM) [7, 9] to evaluate the failure probabilities and formulate the linearly approximated probabilistic constraint in terms of MPFPs, has been developed to solve RBDO problems. The RIA [18] has been further modified to have stable evaluations of reliability indices during the optimization processes; therefore, the failure probabilities can be calculated correctly despite the feasibilities of design points.

Alternatively, the inverse reliability analysis [3], which inverses the CDF of the failure probability to evaluate the performance function value, has been an essential approach in the study of RBDO. Using the inverse reliability analysis, Tu et al. [19] evaluates the target performance function measure at the Most Probable Target Point (MPTP), which is the optimal point of maximizing the performance constraint function under the allowable reliability level in the standard normal design space. For the convex nonlinear constraints, the MPTP has been efficiently determined by the Advanced Mean Value (AMV) method [20]; contrarily, the Conjugate Mean Value (CMV) method [21] has been developed for the concave ones. The Hybrid Mean Value (HMV) method [21] examined the constraint curvature and adaptively utilized the AMV and CMV to determine the MPTPs for convex and concave constraints respectively. The Performance Measure Approach (PMA) method [19, 22-25], which minimizes the cost function subject to the approximated probabilistic constraints formulated in terms of the MPTPs, has been greatly utilized to solve the engineering design problems with design uncertainties [24, 26, 27] because of the high efficiency of finding the MPTPs.

There is a huge diversity of the approximate probabilistic constraint formulations among various RBDO algorithms and have been reviewed in the literature. However, there has never been a general formulation for the RBDO methods and the information about the differences between the linearly approximated probabilistic constraints in these methods has been rather limited. In this paper, a Unified Reliability Formulation (URF) is derived from the linear approximation with allowable reliability level in order to unify the linearly approximated probabilistic constraints in various RBDO algorithms into one general equation. Furthermore, the URFs for some basic RBDO algorithms associated with specific sensitivity analyses are investigated while a comprehensive understanding of the differences between them will be presented. Lin et al. [28] suggested a hybrid approach to select the approximate probabilistic constraints from RIA and PMA during the processes of RBDO. RIA finds the MPFP at the limit state to perform accuracy reliability analysis but requires higher function evaluations than PMA. The one-or-another selection in the hybrid approach was effective however it didn’t take advantage of both algorithms at the same time. Therefore, an Ensemble of Unified Reliability Formulations (EURF) is proposed to consider a group of approximate probabilistic constraints from multiple RBDO algorithms at once in order to cover wider range of reliability analyses than any indusial method. The section 4 introduces the derivation of the general equation of the URF. The section 5 demonstrates the URFs for three basic RBDO algorithms: CCP, RIA and PMA. The section 6 introduces the solution processes of the EURF. The numerical examples are shown in the section 7 and the conclusions are presented in the section 8.

4. The Unified Reliability Formulation for RBDO

In this section, a Unified Reliability Formulation (URF) of the approximate probabilistic constraint is derived to unify the probabilistic formulations from various first-order RBDO algorithms into one general inequality equation.

4.1. Evaluations of Failure Probabilities in RBDO

The RBDO design problem is typically formulated as follows:

$$\underset{\boldsymbol{d}}{Min} \quad z(\boldsymbol{d}) \quad s.t. \quad P\left[g_i(\boldsymbol{X})>0\right] \le P_{f,i} \text{ for } i=1\ldots n \; ; \; \boldsymbol{d}^L \le \boldsymbol{d} \le \boldsymbol{d}^U \tag{1}$$

where $\boldsymbol{X}$ is the vector of random design variables; the expected value $\boldsymbol{d}$ is bounded by the lower limit $\boldsymbol{d}^L$ and the upper limit $\boldsymbol{d}^U$; $z(\boldsymbol{d})$ is the cost function; $g_i(\boldsymbol{X})$ is the i^{th} performance constraint; $P[g_i(\boldsymbol{X})>0]$ is the probability of violating the i^{th} constraint; $P_{f,i}$ is the i^{th} allowable failure probability; and n is the number of constraints.

Mathematically, the probability of the system failure in Eq. (1) can be calculated by an integral of the JPDF $f_i(\boldsymbol{x})$ within the infeasible domain, that is

$$P[g_i(\boldsymbol{X})>0]=\int\underset{g_i(\boldsymbol{X})>0}{\cdots\cdots}\int f_i(\boldsymbol{x})dx_1\cdots dx_N \tag{2}$$

where N is the number of the random variables. It is computationally expensive to evaluate the JPDF and compute the integral in Eq. (2). Instead, several algorithms convert the probabilistic constraints to solvable deterministic inequality equations with respect to $\boldsymbol{d}$ and the RBDO problem can be solved by general optimization solvers. In the following subsections, the deterministic formulation for the probabilistic constraint is first derived from the fundamental aspect of linear approximation with the allowable reliability level and the URF for the RBDO methods is proposed.

4.2. Linear Approximation with Allowable Reliability

Suppose the i^{th} probabilistic constraint function with respect to the mean of the random variable is given by

$$G_i(\boldsymbol{d})\equiv P[g_i(\boldsymbol{X})>0]-P_{f,i}\le 0 \tag{3}$$

The probabilistic function G_i can be linearly approximated at an Allowable Reliability Point (ARP) $\boldsymbol{x}_i^A$ such that the failure probability evaluated at $\boldsymbol{x}_i^A$ equals $P_{f,i}$, yielding the following equation:

$$G_i(\boldsymbol{d})\cong(\boldsymbol{d}-\boldsymbol{x}_i^A)\cdot\nabla_{\boldsymbol{x}}G_i(\boldsymbol{x}_i^A)\le 0 \tag{4}$$

In this paper, the notation of dot product is used for the scalar product, the tensor operating on a vector, and the product of tensors [29]. The Eq. (4) implies that the location of ARP and the sensitivity term $\nabla_{\boldsymbol{x}}G_i(\boldsymbol{x}_i^A)$ are crucial for the formulation of probabilistic constraint. The contour of $\boldsymbol{x}_i^A$ is exactly the boundary of the probabilistic constraint; however, it requires huge computational calculations to find enough points in the design space.

Various linear RBDO methods have utilized different approaches to find the ARP and evaluate the sensitivity at ARP in the Eq. (4). No matter which method is used, the failure probability is evaluated with respect to the gradient vector of the performance constraint at some design point. Therefore, a general gradient-based transformation [30], which converts the multi-dimensional design space to a single-variate space along the gradient direction and scales by the factor of the gradient length, is studied, as in

$$y_i\equiv(\boldsymbol{x}-\boldsymbol{x}_i^G)\cdot\nabla_{\boldsymbol{x}}g_i(\boldsymbol{x}_i^G) \tag{5}$$

where $\boldsymbol{x}_i^G$ is called the Gradient-based Transformation Point (GTP). The ARP is then determined along the i^{th} gradient direction, shown in Figure 1 (a), in order to obtain the approximate probabilistic constraint in Eq. (4). Therefore, the GTP and the PDF of the gradient-based transformed random variable Y_i, illustrated in the subfigure (b), are crucial for the determinations of the ARP and sensitivity of the probabilistic constraint.

Statistically, the reliability level can be quantitatively evaluated by the reliability index [1], which is given by the negative expectation of the performance function measured in the standard normal unit of the performance function. Assuming the random variables are mutually independent and linearly approximating the second moment [8], the allowable reliability index is given by

$$\beta_i(\boldsymbol{x}_i^A)=-\left[g_i(\boldsymbol{x}_i^G)+(\boldsymbol{x}_i^A-\boldsymbol{x}_i^G)\cdot\nabla_{\boldsymbol{x}}g_i(\boldsymbol{x}_i^G)\right]\left\|\boldsymbol{\sigma}\cdot\nabla_{\boldsymbol{x}}g_i(\boldsymbol{x}_i^G)\right\|^{-1}=\beta_{f,i} \tag{6}$$

where the linear approximation of the performance function is expanded at the GTP and $\boldsymbol{\sigma}=\sum_{j=1}^{N}\sigma_j\boldsymbol{e}_j\boldsymbol{e}_j$ is the standard deviation matrix. The constitutive relation between the ARP and the GTP is then derived from Eq. (6):

$$\boldsymbol{x}_i^A=\boldsymbol{x}_i^G-g_i(\boldsymbol{x}_i^G)\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i(\boldsymbol{x}_i^G)\left\|\boldsymbol{\sigma}\cdot\nabla_{\boldsymbol{x}}g_i(\boldsymbol{x}_i^G)\right\|^{-2}-\beta_{f,i}\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i(\boldsymbol{x}_i^G)\left\|\boldsymbol{\sigma}\cdot\nabla_{\boldsymbol{x}}g_i(\boldsymbol{x}_i^G)\right\|^{-1} \tag{7}$$

The Eq. (7) implies that the location of ARP can be determined by the addition of three vectors: the GTP, the reliability measure at GTP along the scaled gradient direction $\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i(\boldsymbol{x}_i^G)\left\|\boldsymbol{\sigma}\cdot\nabla_{\boldsymbol{x}}g_i(\boldsymbol{x}_i^G)\right\|^{-1}$, and the coordinate shift of $\beta_{f,i}$ along the negative scaled gradient direction.

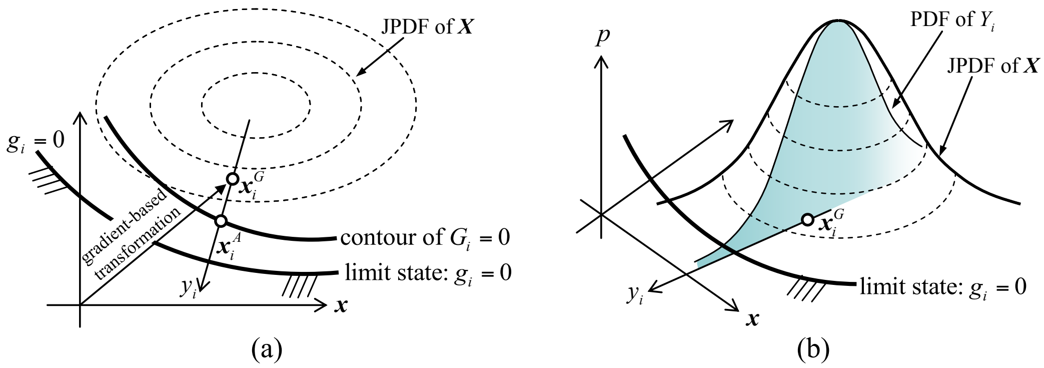

Figure 1: Gradient-based transformation from $\boldsymbol{x}$ -space to y_i -space: (a) a bird's eye view; (b) angular view.

4.3. Sensitivity Analysis

In this subsection, the sensitivity of the probabilistic constraint will be derived based on the relation of ARP and GTP. Using the chain rule, the sensitivity of the probabilistic constraint can be rewritten as

$$\nabla_{\boldsymbol{x}} G_i(\boldsymbol{d}) = \nabla_{y_i} G_i(\boldsymbol{d}) \cdot \nabla_{\boldsymbol{x}} y_i = \nabla_{y_i} G_i(\boldsymbol{d}) \cdot \nabla_{\boldsymbol{x}} g_i(\boldsymbol{x}_i^G) \tag{8}$$

To study the sensitivity term of $\nabla_{y_i} G_i(\boldsymbol{d})$, the probabilistic constraint should be investigated along the gradient-based transformation direction. Approximating the performance function at $\boldsymbol{x}_i^G$, the Eq. (3) is rewritten as follows:

$$G_i(\boldsymbol{d}) \cong P\left[g_i(\boldsymbol{x}_i^G) + (\boldsymbol{X} - \boldsymbol{x}_i^G) \cdot \nabla_{\boldsymbol{x}} g_i(\boldsymbol{x}_i^G) > 0\right] - P_{f,i} \tag{9}$$

Applying the gradient-based transformation to Eq. (9) yields to

$$G_i(\boldsymbol{d}) \cong P\left[Y_i(\boldsymbol{d}) > -g_i(\boldsymbol{x}_i^G)\right] - P_{f,i} \tag{10}$$

where Y_i is the gradient-based transformed random variable. Therefore, the sensitivity of the probabilistic constraint with respect to the gradient-based transformed variable is given by

$$\nabla_{y_i} G_i(\boldsymbol{d}) = \nabla_{y_i} P\left[Y_i(\boldsymbol{d}) > -g_i(\boldsymbol{x}_i^G)\right] = f_{Y_i}\left[-g_i(\boldsymbol{x}_i^G)\right] \tag{11}$$

where f_{Y_i} is the PDF of Y_i. Therefore, the sensitivity term $\nabla_{\boldsymbol{x}} G_i(\boldsymbol{x}_i^A)$ in Eq. (4) is rewritten as

$$\nabla_{\boldsymbol{x}} G_i(\boldsymbol{x}_i^A) = f_{Y_i}\left[-g_i(\boldsymbol{x}_i^G)\right] \nabla_{\boldsymbol{x}} g_i(\boldsymbol{x}_i^G) \tag{12}$$

where $f_{Y_i}\left[-g_i(\boldsymbol{x}_i^G)\right]$ remains positive for the continuous random distribution in the entire design space.

Finally, the URF is defined in the Eq. (13) by the substitutions of Eqs. (7) and (12) to the Eq. (4).

$$\begin{aligned} G_i^{URF}(\boldsymbol{d}) &\equiv G_i(\boldsymbol{d}) \big/ f_{Y_i}\left[-g_i(\boldsymbol{x}_i^G)\right] \\ &\cong \left[\boldsymbol{d} - \boldsymbol{x}_i^G + g_i(\boldsymbol{x}_i^G)\boldsymbol{\sigma}^2 \cdot \nabla_{\boldsymbol{x}} g_i(\boldsymbol{x}_i^G)\left\|\boldsymbol{\sigma} \cdot \nabla_{\boldsymbol{x}} g_i(\boldsymbol{x}_i^G)\right\|^{-2} + \beta_{f,i}\boldsymbol{\sigma}^2 \cdot \nabla_{\boldsymbol{x}} g_i(\boldsymbol{x}_i^G)\left\|\boldsymbol{\sigma} \cdot \nabla_{\boldsymbol{x}} g_i(\boldsymbol{x}_i^G)\right\|^{-1}\right] \cdot \nabla_{\boldsymbol{x}} g_i(\boldsymbol{x}_i^G) \le 0 \end{aligned} \tag{13}$$

which is a general linear approximation of the probabilistic constraint. In the derived URF, the selection of GTP is crucial for the location of the ATP and the gradient direction of the probabilistic constraint. Various RBDO algorithms have considered different GTPs to formulate the linearly approximated probabilistic constraints in this unified format. In the next section, the approximate probabilistic constraints of some basic RBDO methods will be reviewed and the URFs for these methods will be first revealed in the literature.

5. Basic formats of the URFs

The URFs associated with the mean-value, reliability, and inverse reliability analyses have previously been studied. In this section, the basic RBDO methods, which are directly linked to the three reliability aspects, are reviewed and their corresponding URFs are determined.

5.1. Chance Constrained Programming (CCP)

Cornell [1] defined the reliability index β_i to represent the reliability level of the system. Assuming the random variable is normally distributed and transforming the coordinate to the standard normal design space, the failure probability is then approximated by a standard normal CDF of the reliability index [2], which is given as

$$P\left[g_i\left(\boldsymbol{X}\right)>0\right]\cong\Phi\left(-\beta_i\right) \tag{14}$$

Using an inverse transformation of the standard normal CDF [3], the probabilistic constraint in Eq. (14) is then converted to a solvable formulation:

$$-\beta_i\left(\boldsymbol{d}\right)\le-\beta_{f,i} \tag{15}$$

Linearly approximating the $\beta_i\left(\boldsymbol{d}\right)$ at the mean value of the random variable, the Eq. (16) is determined.

$$\left[g_i\left(\boldsymbol{d}^{(k)}\right)+\left(\boldsymbol{d}-\boldsymbol{d}^{(k)}\right)\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{d}^{(k)}\right)\right]\left\|\boldsymbol{\sigma}\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{d}^{(k)}\right)\right\|^{-1}\le-\beta_{f,i} \tag{16}$$

This approach is called the CCP [4, 5]. The approximate probabilistic constraint in Eq. (16) can be easily transform to the following equation:

$$\left[\boldsymbol{d}-\boldsymbol{d}^{(k)}+g_i\left(\boldsymbol{d}^{(k)}\right)\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{d}^{(k)}\right)\left\|\boldsymbol{\sigma}\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{d}^{(k)}\right)\right\|^{-2}+\beta_{f,i}\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{d}^{(k)}\right)\left\|\boldsymbol{\sigma}\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{d}^{(k)}\right)\right\|^{-1}\right]\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{d}^{(k)}\right)\le 0 \tag{17}$$

which is identical to the URF in Eq. (13) using the mean of the k^{th} random variable as the GTP. Therefore, the URF for the mean-value reliability analysis in the CCP has been revealed.

5.2. Reliability Index Approach (RIA)

In RIA [2, 11-15], the reliability index has first been defined as the distance from the point to the failure region measured in standard deviation units [7]. Lin et al. [18] modified the reliability index as follows:

$$\beta_i=\boldsymbol{u}_i^*\cdot\nabla_{\boldsymbol{u}}g_i\left(\boldsymbol{u}_i^*\right)\left\|\nabla_{\boldsymbol{u}}g_i\left(\boldsymbol{u}_i^*\right)\right\|^{-1} \tag{18}$$

In Eq. (18), the MPTP $\boldsymbol{u}_i^*$ is the design point that has the shortest distance from the current design point to the limit state of the i^{th} performance constraint measured in the standard normal unit. Substituting equation reference goes hereEq. (18) to Eq. (15) and linearly approximate $\beta_i\left(\boldsymbol{d}\right)$ at the current deign point, the following approximate probabilistic constraint is obtained:

$$-\boldsymbol{u}_i^*\cdot\nabla_{\boldsymbol{u}}g_i\left(\boldsymbol{u}_i^*\right)\left\|\nabla_{\boldsymbol{u}}g_i\left(\boldsymbol{u}_i^*\right)\right\|^{-1}+\left(\boldsymbol{d}-\boldsymbol{d}^{(k)}\right)\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{u}_i^*\right)\left\|\nabla_{\boldsymbol{u}}g_i\left(\boldsymbol{u}_i^*\right)\right\|^{-1}\le-\beta_{f,i} \tag{19}$$

Transforming the standard normal variables to original design space using the relation of $\boldsymbol{u}=\boldsymbol{\sigma}^{-1}\cdot\left(\boldsymbol{d}-\boldsymbol{d}^{(k)}\right)$, the Eq. (19) can be rewritten as follows:

$$\left[\boldsymbol{d}-\boldsymbol{x}_i^*+\beta_{f,i}\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^*\right)\left\|\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^*\right)\right\|^{-1}\right]\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^*\right)\le 0 \tag{20}$$

where $\boldsymbol{x}_i^*$ is the MPTP in the original design space. Since $g_i\left(\boldsymbol{x}_i^*\right)=0$, the Eq. (20) is identical to the URF in Eq. (13) using the MPTP as the GTP.

5.3. Performance Measure Approach (PMA)

PMA utilized the inverse reliability analysis to evaluate the target performance measure [19] as in

$$\gamma_i\left(\boldsymbol{d}\right)=F_{g_i}^{-1}\left\{1-\Phi\left[-\beta_{f,i}\right]\right\} \tag{21}$$

where $F_{g_i}\left(\gamma\right)$ is the probability of the event of $g_i\left(\boldsymbol{X}\right)\le\gamma$. Since the probabilistic constraint in Eq. (1) can be rewritten as

$$P\left[g_i\left(\boldsymbol{X}\right)>0\right]=1-F_{g_i}\left(0\right)\le\Phi\left[-\beta_{f,i}\right] \tag{22}$$

an approximate probabilistic constraint is obtained by applying Eq. (21) into Eq. (22):

$$\gamma_i\left(\boldsymbol{d}\right)\le 0 \tag{23}$$

The target performance measure is then evaluated at the MPTP $\boldsymbol{u}_i^\#$ and a linear approximate probabilistic constraint is formulated as

$$g_i\left(\boldsymbol{u}_i^\#\right)+\left(\boldsymbol{d}-\boldsymbol{d}^{(k)}\right)\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{u}_i^\#\right)\le 0 \tag{24}$$

where $\boldsymbol{u}_i^\#=\beta_{f,i}\nabla_{\boldsymbol{u}}g_i\left(\boldsymbol{u}_i^\#\right)\left\|\nabla_{\boldsymbol{u}}g_i\left(\boldsymbol{u}_i^\#\right)\right\|^{-1}$ is found toward the direction of most probable target on the $\beta_{f,i}$-sphere centered at the current design point. A dummy term of $\boldsymbol{\sigma}\cdot\left[-\boldsymbol{u}_i^\#+\beta_{f,i}\nabla_{\boldsymbol{u}}g_i\left(\boldsymbol{u}_i^\#\right)\left\|\nabla_{\boldsymbol{u}}g_i\left(\boldsymbol{u}_i^\#\right)\right\|^{-1}\right]$ is added to the

Eq. (24), as in

$$\left[\boldsymbol{d}-\boldsymbol{d}^{(k)}-\boldsymbol{\sigma}\cdot\boldsymbol{u}_i^{\#}+\beta_{f,i}\boldsymbol{\sigma}\cdot\nabla_{\boldsymbol{u}}g_i\left(\boldsymbol{u}_i^{\#}\right)\left\|\nabla_{\boldsymbol{u}}g_i\left(\boldsymbol{u}_i^{\#}\right)\right\|^{-1}\right]\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^{\#}\right)+g_i\left(\boldsymbol{x}_i^{\#}\right)\leq 0 \tag{25}$$

Transforming the standard normal variable to the original design space, the Eq. (25) is then converted to the URF in Eq. (26) which utilizes the MPTP as the GTP.

$$\left[\boldsymbol{d}-\boldsymbol{x}_i^{\#}+g_i\left(\boldsymbol{x}_i^{\#}\right)\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^{\#}\right)\left\|\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^{\#}\right)\right\|^{-2}+\beta_{f,i}\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^{\#}\right)\left\|\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^{\#}\right)\right\|^{-1}\right]\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^{\#}\right)\leq 0 \tag{26}$$

where $\boldsymbol{x}_i^{\#}$ is the MPTP in the original design space.

5.4. Remarks about the URFs for the existing RBDO algorithms

Lin and Gea [31] have compared the solution processes using the linear approximate probabilistic constraints of CCP, RIA, and PMA. The results showed the approximation in CCP is less accurate than the ones in RIA and PMA. The solution process of PMA is the most efficient when HMV is used. RIA guarantees to perform the reliability analysis on the limit state of the performance constraint for continuous random variables. Based on the newly revealed URFs, the major differences between each RBDO algorithms are the selections of the GTPs. To sum up, CCP, RIA, and PMA consider the different GTPs to find the expansion point with the allowable reliability, shown as follows:

$$\boldsymbol{x}_i^{A}=\begin{cases}\boldsymbol{d}^{(k)}-g_i\left(\boldsymbol{d}^{(k)}\right)\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{d}^{(k)}\right)\left\|\boldsymbol{\sigma}\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{d}^{(k)}\right)\right\|^{-2}-\beta_{f,i}\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{d}^{(k)}\right)\left\|\boldsymbol{\sigma}\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{d}^{(k)}\right)\right\|^{-1} & \text{for CCP}\\ \boldsymbol{x}_i^{*}-\beta_{f,i}\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^{*}\right)\left\|\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^{*}\right)\right\|^{-1} & \text{for RIA}\\ \boldsymbol{x}_i^{\#}-g_i\left(\boldsymbol{x}_i^{\#}\right)\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^{\#}\right)\left\|\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^{\#}\right)\right\|^{-2}-\beta_{f,i}\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^{\#}\right)\left\|\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^{\#}\right)\right\|^{-1} & \text{for PMA}\end{cases} \tag{27}$$

In other words, CCP and PMA estimate the design points on the limit state of the performance constraint, denoted as estimated design point (EDP), using the vectors of $\boldsymbol{d}^{(k)}-g_i\left(\boldsymbol{d}^{(k)}\right)\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{d}^{(k)}\right)\left\|\boldsymbol{\sigma}\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{d}^{(k)}\right)\right\|^{-2}$ and $\boldsymbol{x}_i^{\#}-g_i\left(\boldsymbol{x}_i^{\#}\right)\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^{\#}\right)\left\|\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^{\#}\right)\right\|^{-2}$, respectively. Therefore, the accuracy of the reliability analysis increases when the performance measure at the EDP is close to zero, as in Eq. (28).

$$g_i\left[\boldsymbol{x}_i^{G}-g_i\left(\boldsymbol{x}_i^{G}\right)\boldsymbol{\sigma}^2\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^{G}\right)\left\|\boldsymbol{\sigma}\cdot\nabla_{\boldsymbol{x}}g_i\left(\boldsymbol{x}_i^{G}\right)\right\|^{-2}\right]\approx 0 \tag{28}$$

In the following section, the well-fit functions are found based on the checking criteria in Eq. (28) and considered simultaneously during the optimization processes.

6. EURF

In this paper, an Ensemble of Unified Reliability Formulations (EURF) is proposed to consider a group of the well-fit approximate probabilistic constraints from multiple RBDO algorithms at once in order to cover wider range of reliability analyses than any indusial method. The general optimization formulation of the EURF is defined as follows:

$$\begin{aligned}&\underset{\boldsymbol{d}}{Min} && z\left(\boldsymbol{d}\right)\\ &s.t. && \left\{\begin{array}{ll}\bigcup_{m=S_i}\left[G_{i,m}^{URF}\left(\boldsymbol{d}\right)\leq 0\right] & \text{for convex } g_i\\ \bigcap_{m=S_i}\left[G_{i,m}^{URF}\left(\boldsymbol{d}\right)\leq 0\right] & \text{for concave } g_i\end{array}\right\} \quad i=1\ldots n\\ &&& \boldsymbol{d}^{L}\leq\boldsymbol{d}\leq\boldsymbol{d}^{U}\end{aligned} \tag{29}$$

where S_i is the set of well-fit approximate probabilistic constraints from multiple RBDO algorithms. In our implementation, the indices for CCP, RIA, and PMA are given as 1, 2, and 3 respectively. In Eq. (29), the symbols of $\bigcup$ and $\bigcap$ represent the operations of union and intersection respectively. From section 5, each RBDO algorithm performs a unique reliability analysis to determine the ARP and formulate the approximate probabilistic constraint regarding to different selections of the GTPs. The probabilistic constraints generated by each RBDO algorithm are mostly different from each during the optimization processes. In the common practices, only one formulation is chosen from multiple algorithms based on the designers' preferences [31]. The Eq. (29) represents a novel approach to take advantage of the linear approximate probabilistic constraints from various RBDO algorithms in terms of considering every well-fit URFs at once during the optimization processes.

From Eq. (28), the goodness of fit of the approximate probabilistic constraint is determined based on the closeness of the EDP to the limit state of the performance function, illustrated by the highlighted area in Figure 2. The probabilistic constraints from a benchmark mathematical problem in the literature [22, 23, 32] , as in Eqs. (30) and (31), are considered in the subfigures (a) and (b), respectively.

$$P\left[g_a\left(\boldsymbol{X}\right)=1-\left(X_1^2X_2\right)/20>0\right]\leq 0.13\% \tag{30}$$

$$P\left[g_b\left(\boldsymbol{X}\right)=1-\left(X_1+X_2-5\right)^2/30-\left(X_1-X_2-12\right)^2/120>0\right]\leq 0.13\% \tag{31}$$

The location of $\boldsymbol{d}^{(k)}$ is $[5,5]^T$ and the standard deviations are $\left[\sigma_1,\sigma_2\right]=\left[0.3,0.3\right]$. There should not be exact criteria for the closeness of the EDPs. For the maximum accuracy of the reliability analysis, the allowable range of the well-fit area is narrowed to zero and the URF of RIA will be utilized. The relaxation of the range of allowable closeness between the EDP and the limit state enables the consideration of more URFs from other RBDO algorithms, which is beneficial for the realization of the shape of the true probabilistic constraint. For example, Figure 2 (a) shows that the EDPs of CCP and PMA are outside of the well-fit area. The EURF, which therefore includes only the URF of RIA in Eq. (20), is estimated as a linear probabilistic constraint. Furthermore, the subfigure (b) shows that only the EDP of CCP is outside of the well-fit area. In this circumstance, both the URFs from RIA and PMA are considered in the ensemble of constraints in order to have better approximation of the nonlinear behavior of the probabilistic constraint. From Eq. (29), the union operation is utilized since the limit state is a convex function near the investigated area.

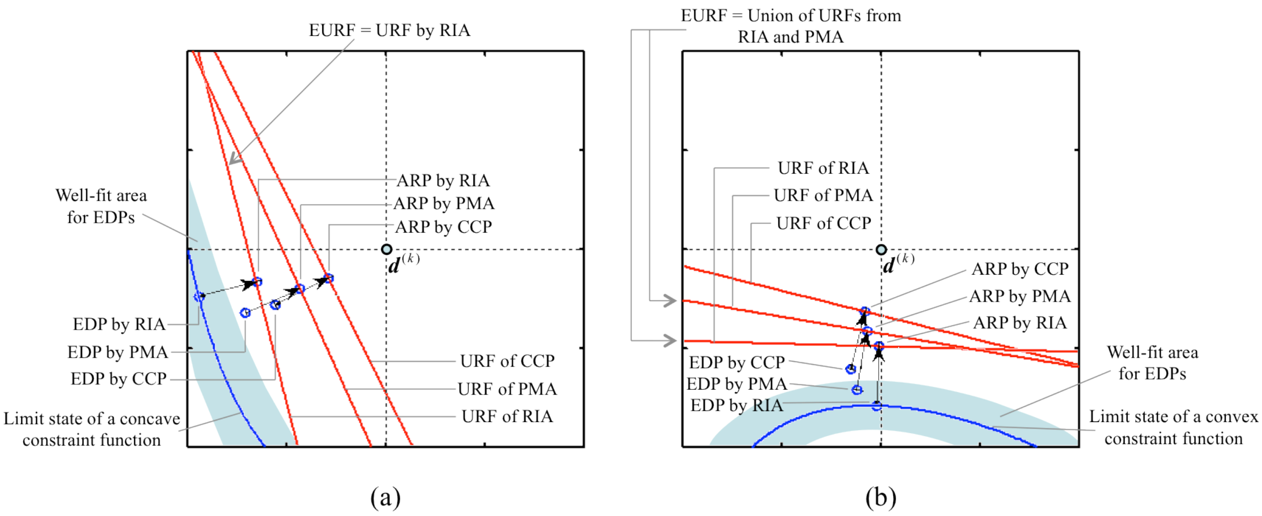

Figure 2: Linear approximate probabilistic constraints of CCP, RIA, and PMA for (a) the concave constraint and (b) the convex constraint.

The convexity of the limit state of the performance constraint near the investigated location is crucial for the selections of probabilistic constraints in the EURF, previously presented in Eq. (29). The constraint convexity can be determined by the calculation in terms of the information from any two well-fit URFs, as shown in Figure 3. The angle θ_{ij} represents the angle between the vectors from the i^{th} EDP to the i^{th} ARP and the j^{th} EDP. When $\theta_{ij}<90°$, the limit state is concave near the i^{th} and j^{th} EDPs. When $\theta_{ij}>90°$, the limit state is convex. The situation of $\theta_{ij}=90°$ occurs when the limit state function is linear; otherwise, it is rare in the numerical implementation for nonlinear constraints. The situation of $\theta_{ij}\approx 90°$ may happen when the investigated EDPs are very close to each other, yielding minimal differences between the investigated URFs. In our implementation, only the URF with the most accurate reliability analysis is chosen in replace of the EURF. In the circumstances of disagreed convexity analysis (such as more than two well-fit URFs near a saddle point), only the URF with the most accurate reliability analysis is considered.

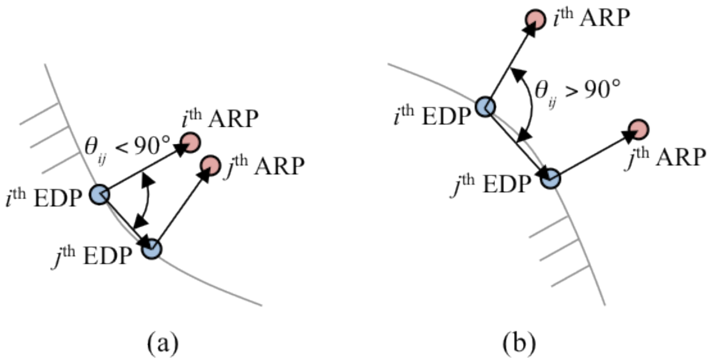

Figure 3: Convexity investigation for (a) the concave constraint and (b) the convex constraint.

7. Numerical Examples

In this section, two benchmark mathematical problems are studied. The problems will be solved by the individual RBDO methods, including CCP, RIA, and PMA, as well as the proposed EURF. The Monte Carlo Simulations will be utilized to confirm the correctness of the optimal solutions. The required numerical performances will be compared to demonstrate the efficiency of each method.

7.1. Example 1: Benchmark Mathematical Problem

The same benchmark mathematical problem in the section 6 is considered as the first numerical example. The optimization problem formulation is given as follows:

$$\begin{aligned} \underset{d}{Min} \quad & z(\boldsymbol{d}) = d_1 + d_2 \\ s.t. \quad & P\left[g_1(\boldsymbol{X}) = 1 - \left(X_1^2 X_2\right)/20 > 0\right] \le 0.13\% \\ & P\left[g_2(\boldsymbol{X}) = 1 - \left(X_1 + X_2 - 5\right)^2/30 - \left(X_1 - X_2 - 12\right)^2/120 > 0\right] \le 0.13\% \\ & P\left[g_3(\boldsymbol{X}) = 1 - 80/\left(X_1^2 + 8X_2 + 5\right) > 0\right] \le 0.13\% \\ & 0.1 \le d_1, d_2 \le 10 \end{aligned} \tag{32}$$

The random variables are assumed normally distributed and mutually independent. The location of initial design is $[5,5]^T$ and the standard deviations are $\left[\sigma_1, \sigma_2\right] = \left[0.3, 0.3\right]$. In our implementation, the criteria of allowable closeness of the well-fit area in Eq. (33) are given by

$$\left| g_i\left[\boldsymbol{x}_i^G - g_i\left(\boldsymbol{x}_i^G\right)\boldsymbol{\sigma}^2 \cdot \nabla_{\boldsymbol{x}} g_i\left(\boldsymbol{x}_i^G\right)\left\|\boldsymbol{\sigma} \cdot \nabla_{\boldsymbol{x}} g_i\left(\boldsymbol{x}_i^G\right)\right\|^{-2}\right]\right| \le \alpha \left\|\boldsymbol{\sigma} \cdot \nabla_{\boldsymbol{x}} g_i\left(\boldsymbol{x}_i^G\right)\right\| \tag{33}$$

where $\alpha = 0.2$ is defined so that no disagreement of the constraint convexity is found. The results are listed in Table 1. CCP requires the highest iteration number to converge and it is too conservative as none of the failure probabilities (denoted by pf_i) approach to the allowable limit. RIA, PMA, and EURF all converge at the same optimal solution (denoted by $\boldsymbol{d}^{opt}$) using 4 iterations. It is expected that the function evaluations of EURF are almost double of the ones of RIA and PMA. It is guaranteed the iteration of EURF won't exceed RIA and PMA.

7.2. Example 2: Benchmark Highly Nonlinear Mathematical Problem

The next example is a highly nonlinear mathematical problem [33], which is given by

$$\begin{aligned} \underset{d}{Min} \quad & z(\boldsymbol{d}) = -\left(d_1 + d_2 - 10\right)^2/30 - \left(d_1 - d_2 + 10\right)^2/120 \\ s.t. \quad & P\left[g_1(\boldsymbol{X}) = 1 - \left(X_1^2 X_2\right)/20 > 0\right] \le 0.13\% \\ & P\left[\begin{aligned} g_2(\boldsymbol{X}) = -1 &+ \left(0.9063X_1 + 0.4226X_2 - 6\right)^2 + \left(0.9063X_1 + 0.4226X_2 - 6\right)^3 \\ & -0.6\left(0.9063X_1 + 0.4226X_2 - 6\right)^4 - \left(-0.4226X_1 + 0.9063X_2\right) > 0 \end{aligned}\right] \le 0.13\% \\ & P\left[g_3(\boldsymbol{X}) = 1 - 80/\left(X_1^2 + 8X_2 + 5\right) > 0\right] \le 0.13\% \\ & 0.1 \le d_1, d_2 \le 10 \end{aligned} \tag{34}$$

The random variables are assumed normally distributed and mutually independent. The location of initial design is $[3.5, 4]^T$ and the standard deviations are $\left[\sigma_1, \sigma_2\right] = \left[0.2, 0.2\right]$. Only the accurate reliability analyses in RIA and PMA are considered in this highly nonlinear mathematical problem. There will be no disagreement in the convexity check when only two URFs are considered. Therefore, α is relaxed to 0.5 in our implementation. Furthermore, the convexity is only checked when the investigated EDPs are close to each other. The maximum allowable distance between EDPs is defined as $\beta_{f,i} \times \min(\sigma)$. The results are listed in the Table 1. RIA, PMA and EURF all are capable of finding the correct optimal solution while EURF requires the fewest function evaluations to converge at the solution. In the current literature, it is difficult to find an existing method that requires less iteration to find optimal solutions than both RIA and PMA. Moreover, the EURF will be most beneficial for the real-world applications where each function evaluation is cheap and fast.

Table 1. Results of the Numerical Examples.

Example	Method	$[d_1^{opt}, d_2^{opt}]$	Iteration	Function Evaluations	$[pf_1, pf_2, pf_3]$
1	CCP	[3.6364, 3.4472]	8	72	[0.0066%, 0.0422%, 0%]
	RIA	[3.4390, 3.2866]	4	227	[0.1463%, 0.1152%, 0%]
	PMA	[3.4391, 3.2866]	4	231	[0.1547%, 0.1150%, 0%]
	EURF[a]	[3.4391, 3.2866]	4	568	[0.1451%, 0.1137%, 0%]
2	RIA	[4.6716, 1.5684]	5	319	[0.1504%, 0.0887%, 0%]
	PMA	[4.6716, 1.5684]	7	474	[0.1514%, 0.0849%, 0%]
	EURF[b]	[4.6709, 1.5690]	4	618	[0.1462%, 0.0810%, 0%]

a: The URFs from CCP, RIA, and PMA are considered simultaneously.
b: The URFs from RIA and PMA are considered simultaneously.

8. Conclusions

In the Reliability-Based Design Optimization (RBDO), the probabilistic constraints have been linearly approximated with various approaches of reliability analysis but there is a huge diversity of the approximate constraints in the existing methods. In this paper, the Unified Reliability Formulation (URF) is derived from the linear approximation with allowable reliability level and is utilized to unify the probabilistic constraints in various RBDO algorithms into one general equation. The determinations of the Allowable Reliability and the Gradient-based Transformation Points are crucial for the URF. The URFs for three basic RBDO algorithms have newly been revealed: Chance Constrained Programming (CCP), Reliability Index Approach (RIA), and Performance Measure Approach (PMA) utilize the mean point, the most probable failure points, and the most probable target points as the GTPs, respectively. An Ensemble of Unified Reliability Formulations (EURF) has been proposed to take advantage of the linear approximate probabilistic constraints from each RBDO algorithm. The goodness of fit of the probabilistic constraint is first verified. The union and intersection of the well-fit probabilistic constraints are then considered for concave and convex limit states, respectively. Two benchmark mathematical problems have been studied to show the numerical performances of the proposed method. As a result, EURF is capable of finding the correct optimal solutions. It is guaranteed the iteration of EURF won't exceed RIA and PMA. It is beneficial for some real-world applications that EURF may require less iteration to find the optimal solutions than RIA and PMA. However, EURF requires more function evaluations than any individual methods. Some other URFs are currently under investigation in Chung Yuan Christian University to improve the efficiency of EURF.

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