Adaptive reduction of random variables using global sensitivity in reliability-based optimisation

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Abstract: This paper presents an efficient shape optimisation technique based on Stochastic Response Surfaces (SRS) and adaptive reduction of random variables using global sensitivity information. The SRS is a polynomial chaos expansion that uses Hermite polynomial bases and provides a closed form solution of the model output from a significantly lower number of model simulations than those required by conventional methods such as the Monte Carlo simulations and Latin Hypercube sampling. Random variables are adaptively fixed before constructing the SRS if their corresponding Global Sensitivity Indices (GSI) calculated using the low-order SRS are below a certain threshold. It has been shown that the GSI can be calculated analytically because the SRS employs the Hermite polynomials as bases. Using SRS and adaptive reduction of random variables, reliability-based optimisation problems are solved with a significant reduction in computational cost. The efficiency and convergence of the proposed approach is demonstrated using a benchmark case and an industrial Reliability-Based Design Optimisation (RBDO) problem.

Keywords: reliability-based optimisation; reliability analysis; uncertainty; global sensitivity; sensitivity analysis; stochastic response surface; polynomial chaos.

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1 Introduction

Uncertainty in the design parameters makes shape optimisation of structural systems a computationally expensive task due to the significant number of analyses required by traditional methods. Critical issues for overcoming these difficulties are those related to uncertainty characterisation, uncertainty propagation, ranking of design variables and efficient optimisation algorithms. Traditional approaches for these tasks often fail to meet constraints (computational resources, cost, time, etc.) typically present in industrial environments.

In particular, Reliability-Based Design Optimisation (RBDO) involving a computationally intensive model has been limited by the relatively high number of required analyses for uncertainty propagation during the optimisation process. While there have been progresses addressing this issue, such as more efficient moment-based optimisation algorithms (e.g. RIA, (Tu et al., 1999) PMA (Tu et al., 1999)) and the construction of Stochastic Response Surfaces (SRS) for uncertainty propagation, (Kim et al., 2006) the possibility of reducing the number of analyses by systematically fixing unessential random variables throughout the optimisation process has not been fully explored. When the contribution of a random variable to the variability of the output is negligible, it can be treated as a deterministic variable with its mean value. The issues are how the contribution of a random variable can be calculated effectively and what the threshold is so that it can be considered as a deterministic variable.

In this paper, to avoid the shortcomings of the conventional moment-based methods (FORM or SORM), the Monte Carlo Simulations, (Doll and Freeman, 1986) Latin

Hypercube Sampling, (Iman and Conover, 1980) and those associated with the use of the SRS, the following two sensitivities are utilised:

1 local sensitivity information is also used to reduce the number of sampling points and

2 Global Sensitivity Indices (GSI) are calculated to decide whether to fix random variables whose contribution to the output variability is less than a certain threshold.

Local sensitivity is obtained by differentiating the variational equation of mechanics at a given design point, whereas global sensitivity is calculated by integrating the contribution of a random variable over the entire domain.

The objective of this paper is to fix unessential random variables during reliability-based optimisation process. Unessential random variables are identified and fixed based on their contribution to the variability of model output (global sensitivities). To make the procedure efficient, a low-order SRS with all random variables is used to calculate global sensitivities, whereas a high-order SRS with reduced random variables is used to evaluate the probability of failure of the system. As an unessential variable at one design may become essential at a different design, the global sensitivities are evaluated at each design and different sets of essential random variables can be selected.

This paper is structured as follows: Section 2 illustrates the proposed procedure of RBDO with adaptive reduction of random variables. Section 3 describes the uncertainty characterisation of random variables, and the uncertainty propagation to the output using SRS. Section 4 presents the procedure to compute GSI to fix unessential random variables during the construction of the SRS. An RBDO problem is formulated and the results obtained using the proposed approach are the subject of Section 5, followed by numerical examples in Section 6.

2 Overview of the solution approach

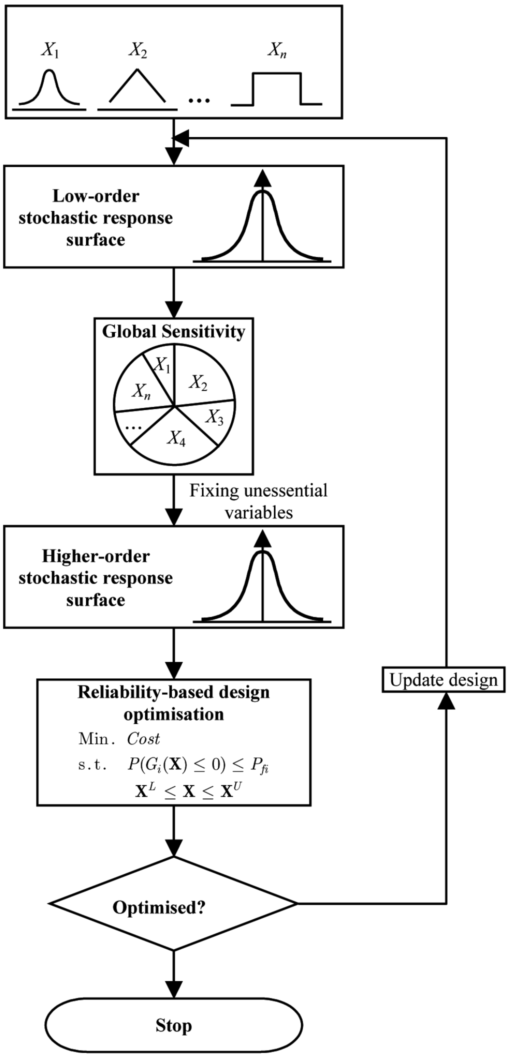
With reference to Figure 1, the proposed approach for RBDO initially constructs a low-order SRS using all random variables, and adaptively reduces them depending on the values of their corresponding GSI. GSI are the contributions of random variables to the variance of the model output. GSI are calculated using a variance-based method (Homma and Saltelli, 1996; Saltelli et al., 1999; Sobol, 1993) – a rigorous and theoretically sound approach for global sensitivity. To facilitate the effective calculation of the GSI, a low-order SRS is first constructed using all random variables. The use of Hermite polynomial bases in SRS makes it possible to calculate GSI analytically. If the GSI of a random variable is smaller than a certain threshold, it is considered as a deterministic one with its mean value assigned to it. Using the reduced number of random variables, a high-order SRS is constructed from which the reliability of the performance function is evaluated. The accuracy of SRS and associated reliability are compared between two approaches:

1 when all random variables are used and

2 when a reduced number of random variables are used based on the GSI.

The proposed adaptive reduction scheme is applied to the RBDO problem.

Figure 1 Adaptive reduction of unessential random design variables using global sensitivity indices in RBDO. A low-order SRS is used for global sensitivity analysis, while a high-order SRS is used to evaluate the reliability of the system



3 Uncertainty quantification

Uncertainty quantification can be decomposed in three fundamental steps:

1 uncertainty characterisation of model inputs

2 propagation of uncertainty and

3 uncertainty management/decision making.

First two steps are explained in this section, while the last step will be incorporated with global sensitivity of model input to output.

The uncertainty in model inputs is represented in terms of standardised normal random variables with mean 0 and variance equal to 1. The selection is supported by the fact that they are widely used and well-behaved. For other types of random variables, an appropriate transformation must be employed. We will assume that the model inputs are independent so each one is expressed directly as a function of the standard normal random variable through a proper transformation. Devroye (1986) presents the required transformation techniques and approximations for a variety of probability distributions. More arbitrary probability distributions can be approximated using algebraic manipulations or by series of expansions. Transforming to standard normal random variables is important because it maintains the orthogonal property of bases in approximation.

The uncertainty propagation is based on constructing SRS (polynomial chaos expansion). The stochastic response surface (Isukapalli et al., 1998) can be viewed as an extension of classical deterministic response (Box and Draper, 1987) surfaces for model outputs constructed using uncertain inputs and performance data collected at heuristically selected collocation points. The stochastic response surface takes into account the probabilistic distribution of input variables in approximation. The polynomial expansion uses Hermite polynomial bases for the space of square-integrable Probability Density Functions (PDF) and provides a closed form solution of the model output from a significanty lower number of simulations than those required by conventional methods such as the Monte Carlo simulations and Latin Hypercube sampling.

Let n be the number of random variables and p be the order of polynomial. The model output can then be expressed in terms of standard normal random variables $\mathbf{u} = \{u_1, u_2, \ldots, u_n\}^{\mathrm{T}}$ as:

$$G^p = a_0^p + \sum_{i=1}^{n} a_i^p \Gamma_1(u_i) + \sum_{i=1}^{n}\sum_{j=1}^{i} a_{ij}^p \Gamma_2(u_i, u_j) + \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} a_{ijk}^p \Gamma_3(u_i, u_j, u_k) + \cdots \qquad (1)$$

where G^p is the approximated model output, the $a_i^p, a_{ij}^p, \ldots$ are deterministic coefficients to be estimated and the $\Gamma_p(u_i, \ldots, u_p)$ are multidimensional Hermite polynomials of degree p given by:

$$\Gamma_p(u_i, \ldots, u_p) = (-1)^p e^{1/2\mathbf{u}^{\mathrm{T}}\mathbf{u}} \frac{\partial^p}{\partial u_i \cdots \partial u_p} e^{-1/2\mathbf{u}^{\mathrm{T}}\mathbf{u}} \qquad (2)$$

where $\mathbf{u}$ is a vector of p independent and identically distributed normal random variables selected among the n random variables that represent the model input uncertainties. Equation (1) is also called a polynomial chaos expansion (Ghanem and Spanos, 1991). The Hermite polynomials $\Gamma_p(u_i, \ldots, u_p)$ are set of orthogonal polynomials with weighting function $e^{-u^2/2}$, which has the same form with the PDF of standard normal random variables. In this paper, a modified version of Hermite polynomial (Abramowitz and Stegun, 1972) is used. The first four terms are u, u^2-1, u^3-3u and u^4-6u^2+3, when a single random variable is involved. The use of the Hermite polynomials has two purposes:

1 they are used to determine the sampling points and

2 they are used as bases for polynomial approximation.

The coefficients in the polynomial chaos expansion are calculated as those providing the best fit (least squares solution), considering a sample of input/output pairs. As all inputs are represented using standard normal random variables, more accurate estimates for the coefficients can be expected if the probability distribution of the u_i's is considered. Hence, a set of points near the high probability region is heuristically selected among the roots of the one-order higher polynomial under restrictions of symmetry and closeness to the mean. In general, the approximation accuracy increases with the order of the polynomial and should be selected reflecting the accuracy needs and computational constraints. In addition, the approximation in Equation (1) includes robust coefficients; that is, their values are not significantly affected when constructing higher order polynomials (Kim et al., 2006). In structural applications, the SRS has been applied to tolerance analysis (Anile et al., 2003; Kim et al., 2003b) and RBDO (Choi et al., 2003; Kim et al., 2006).

The number of model simulations required to construct the SRS could be reduced when local sensitivity information is available. The issue is how efficiently the local sensitivity information can be calculated. If the global finite difference method is employed, there is no advantage in using sensitivity information because each sensitivity calculation requires additional analyses. Recently, Isukapalli et al. (2000) used an automatic differentiation program (ADIFOR (Bischof et al., 1996)) to calculate the local sensitivity of the model output with respect to random variables and used them to construct a stochastic response surface. Their results showed that local sensitivity information can significantly reduce the number of sampling points required. However, the computational cost of the automatic differentiation is often higher than that of direct analysis (Carle et al., 1998).

In contrast, when the finite element method is used, as discussed by van Keulen et al. (2005), design sensitivity analysis can provide a very efficient tool for calculating gradient information because the sensitivity equation uses the same coefficient matrix that is already factorised from the original analysis. In many finite element-based structural analyses, the discrete system is often represented using a matrix equation as

$$[\mathbf{K}]\{\mathbf{D}\} = \{\mathbf{F}\} \tag{3}$$

where $[\mathbf{K}]$ is the stiffness matrix, $\{\mathbf{F}\}$ is the load vector and $\{\mathbf{D}\}$ is the nodal solution. The model output in Equation (1) can be expressed as a function of the nodal solution. Thus, the local sensitivity of the model output can be easily calculated if the local sensitivity of the nodal solution is available. When design parameters are defined, the matrix Equation (3) can be differentiated with respect to the design parameter d_i to obtain the following design sensitivity equation:

$$[\mathbf{K}]\left\{\frac{\partial \mathbf{D}}{\partial d_i}\right\} = \left\{\frac{\partial \mathbf{F}}{\partial d_i}\right\} - \left[\frac{\partial \mathbf{K}}{\partial d_i}\right]\{\mathbf{D}\} \tag{4}$$

The above equation can be solved inexpensively because the matrix $[\mathbf{K}]$ is already factorised when solving Equation (3). The computational cost of sensitivity analysis is less than 20% of the original analysis cost. The efficiency of the uncertainty propagation approach is critical to RBDO (uncertainty management), as at each design cycle an updated version of the PDF for the constraint function (related to model outputs) is required. A rigorous development of design sensitivity analysis can be found in reference

Choi and Kim (2004a) for linear problems and Choi and Kim (2004b) for non-linear problems.

Once the SRS of performance function is obtained, the reliability of the performance function can be obtained through the Monte Carlo simulation performed on the SRS. As an analytical expression for the performance function is available, the computational cost of Monte Carlo simulation is not significant.

4 GSI and an adaptive approach for fixing unessential variables

To reduce the number of simulations required to construct the SRS even further, unessential random variables are fixed during the construction of the SRS. A random variable is considered unessential (and hence it is fixed) if its contribution to the variance of the model output is below a given threshold. GSI considering only main factors are calculated to quantify the model input contributions to the output variance hence establishing which factors influence the model prediction the most so that:

1 resources can be focused to reduce or account for uncertainty where it is most appropriate or

2 unessential variables can be fixed without significantly affecting the output variability.

The latter application is the one of interest in the context of this paper.

Variance-based methods are the most rigorous and theoretically sound approaches for global sensitivity calculations (Homma and Saltelli, 1996; Saltelli et al., 1999: Sobol, 1993). This section describes the fundamentals of the variance-based approach and illustrates how the polynomial chaos expansions are particularly suited for this task.

4.1 Variance-based global sensitivity analysis

The variance-based methods

1 decompose the model output variance as the sum of partial variances and then

2 establish the relative contribution of each random variable (global sensitivity indexes) to the model output variance.

To accomplish step (i), the model output is decomposed as a linear combination of functions of increasing dimensionality as described by the following expression:

$$f(\mathrm{x}) = a_0 + \sum_{i=1}^{n} a_i f_i\left(x_i\right) + \sum_{i=1}^{n}\sum_{j>i}^{n} a_{ij} f_{ij}\left(x_i, x_j\right) + \cdots + a_{12\ldots n} f_{12\ldots n}\left(x_1, x_2, \ldots, x_n\right) \tag{5}$$

subject to the restriction that the integral of the weighted product of any two different functions vanishes. Formally,

$$\int \cdots \int p(\mathbf{x}) f_{i_1,\ldots,i_s}(x_{i_1},\ldots,x_{i_s}) f_{j_1,\ldots,j_s}(x_{j_1},\ldots,x_{j_s}) d\mathbf{x} = 0 \quad \text{for} \quad i_1,\ldots,i_s \neq j_1,\ldots,j_s \tag{6}$$

where $p(\mathbf{x})$ is the joint PDF of input random variable $\mathbf{x}$. If, for example, the weighting function is the uniform distribution for the random variables or the Gaussian probability distribution, the functions of interest can be shown as Legendre and Hermite orthogonal polynomials, respectively.

The model output variance can now be calculated using a well-known result in statistics. The result establishes that the variance of the linear combination of random variables (U_i) can be expressed as:

$$V\left(b_0+\sum_{i=1}^{n} b_i U_i\right)=\sum_{i=1}^{n} b_i^2 V\left(U_i\right)+2\sum_{i=1}^{n}\sum_{j>i}^{n}\mathrm{COV}\left(U_i, U_j\right) \tag{7}$$

Hence, the model output variance can be shown as:

$$V(f)=\sum_{i=1}^{n} a_i^2 V\left(f_i\right)+\sum_{i=1}^{n}\sum_{j>i}^{n} a_{ij}^2 V\left(f_{ij}\right)+\cdots+a_{12\ldots n}^2 V\left(f_{12\ldots n}\right) \tag{8}$$

where the terms represent partial variances and each $V(\cdot)$ may be found by definition as:

$$V(f)=\int\left[f(x)-E\left(f(x)\right)\right]^2 p(x)\,dx \tag{9}$$

In the above formula, $f(x)$ represents the function under consideration and the symbol $E(\cdot)$ denotes expected value.

Of interest here is the global sensitivity index S_i considering only main factor associated with each of the random variables which is represented by Equation (8):

$$S_i=\frac{a_i^2\, V\left(f_i\left(x_i\right)\right)}{V(f)} \tag{10}$$

By comparing Equations (1) and (5), a linear polynomial chaos expansion is enough in obtaining the GSI considering only main factors. The current algorithm with linear polynomial chaos expansion for the reduction of random variables could be modified to use a sensitivity index that accounts for interactions. These interactions will only appear in higher order polynomial chaos expansions. The choice of a non-linear polynomial chaos expansion would reduce the computational efficiency of the proposed approach with unclear significant advantages.

As stated at the beginning of this section, once the GSI are calculated, factors that have the least influence on the model variance (unessential variables) can be identified and eventually held fixed without significantly affecting the output variability. The procedure is adaptive because the GSI are calculated at each design iteration and as a result different sets of random variables may be fixed throughout the RBDO process.

4.2 Global sensitivity analysis using polynomial chaos expansion

The polynomial chaos expansion is particularly suited for computing GSI. Firstly, the model output is already decomposed as a sum of functions of increasing dimensionality. In addition, the functions are orthogonal with respect to the Gaussian measure (Hermite polynomials) and the variance of the bases are analytically available. For example, the functions associated with a two dimensional chaos expansion of order 2 and the corresponding variances are shown in Table 1.

Table 1 Variances of the Hermite bases up to the second order

Function f	*V(f)*
X_1	1
X_2	1
$X_1^2 - 1$	2
$X_1 X_2$	1
$X_2^2 - 1$	2

Secondly, given the polynomial chaos expansion (i.e. the coefficients of the linear combination of Hermite polynomials), the model output variance and GSI can be easily computed using Equations (7) and (10), respectively. In both equations, the variances $V(\cdot)$, are readily available for polynomial chaos expansions of arbitrary order and number of variables.

5 Reliability-based optimisation

To illustrate and evaluate the proposed approach, a simple formulation of the more general RBDO problem (Chandu and Grandhi, 1995; Enevoldsen and Sorensen, 1994; Grandhi and Wang, 1998; Wu and Wang, 1996) is discussed. The cost function is assumed to be easily evaluated using the design variables and the constraints are defined using the probability of failure of the performance functions. Specifically, consider the following form of the RBDO problem:

$$\begin{aligned} &\text{Minimise} && c(\mathbf{d}) \\ &\text{subject to} && P\left(G_j(\mathbf{x}) < 0\right) \le P_{f,j}, \quad j = 1,2,\ldots,np \\ & && \mathbf{d}_L \le \mathbf{d} \le \mathbf{d}_U \end{aligned} \tag{11}$$

with $\mathbf{x} = [x_i]^{\mathrm{T}}$ ($i = 1, 2,\ldots, n$) being the vector of random variables. Each random variables are assumed to be normally distributed, that is, $x_i \sim N(\mu_i, \sigma_i^2)$, with mean μ_i and SD σ_i. In Equation (11), $\mathbf{d} = [d_i]^{\mathrm{T}} = [\mu_i]^{\mathrm{T}}$ represents the design variables chosen as the mean values of $\mathbf{x}$, and $c(\mathbf{d})$ identifies the cost function. The system performance criteria are described by the performance functions $G_j(\mathbf{x})$ such that the system fails if $G_j(\mathbf{x}) < 0$. Each $G_j(\mathbf{x})$ is characterised by its cumulative distribution function $F_G(g)$:

$$F_{G_j}(g) = P\left(G_j(\mathbf{x}) < g\right) = \int_{G_j(\mathbf{x})<g} \cdots \int f_{\mathrm{X}}(\mathbf{x})\, dx_1 \cdots dx_n \tag{12}$$

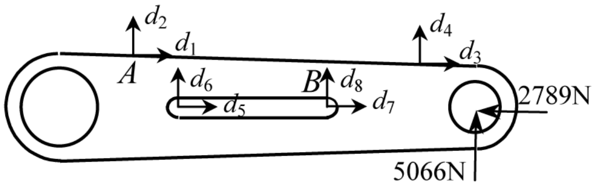
where $f_{\mathbf{X}}(\mathbf{x})$ is the joint PDF of all random system parameters and g is the probabilistic performance measure. The reliability analysis of the performance function requires evaluating the non-decreasing $F_G(g) \sim g$ relationship, (Tu et al., 1999) which is performed in the probability integration domain bounded by the system parameter tolerance limits. As the probability integration domain is in general complicated, many approximation methods (FORM or SORM) are often used. In this paper, the probability of failure estimated using the polynomial chaos expansion and Monte Carlo simulation is used for evaluating reliability constraints hence providing better approximations than traditional linearisation and thus significantly improving the rate of convergence of RBDO. Once the cost and constraint functions are evaluated, the optimisation problem in Equation (11) can be solved using conventional mathematical programming techniques.

6 Numerical examples

6.1 Stochastic response surface for the torque-arm model

The use of the proposed polynomial chaos expansion for the uncertainty propagation and reduction of unessential random variables is demonstrated using a reliability-based optimisation of a structural component. Consider the torque-arm model depicted in Figure 2 (Kim et al., 2003a). The locations of boundary curves have uncertainties due to manufacturing processes. Thus, the relative locations of corner points of the boundary curves are defined as random variables.

Figure 2 Shape design parameters for the torque-arm model. Design parameters are the mean values of corner coordinates of boundary curves. Due to manufacturing processes, the coordinates are represented as normal random variables with a SD of 0.1



For simplicity, we assumed that all random variables exhibit a normal distribution with mean 0 and SD equal to 0.1; that is, $\mathbf{x} \sim N(\mu, 0.1^2)$. The mean values of these random variables are chosen as design parameters, whereas the SD remains constant during the design process.

As illustrated in Figure 2, the initial model consists of eight design parameters. For example, design parameter d_1 is the mean of the relative location of point A in the x-direction. To show how the SRS is constructed and the PDF of the model output is calculated, we choose the three design parameters (d_2, d_6 and d_8) that most significantly contribute to the stress performance at points A and B.

A meshfree method (Kim et al., 2003a) is employed to solve the structural response. In the initial design, the maximum stress of 305 MPa occurs at location A. For reliability analysis, the stress limit is established to be 800 MPa. In the reliability analysis, the performance function is defined such that $G \le 0$ is considered failure. Thus, in the case of stress constraints, the following performance function is defined:

$$G(\mathrm{x}) := \sigma_{\max} - \sigma_A(\mathbf{x}) \tag{13}$$

where $\sigma_{\max}$ is the maximum allowed equivalent stress and σ_A is the stress at location A.

Before constructing the stochastic response surface, it is important to transform the random variables $\{x_i(d_i, 0.1)\}$ into standard random distributions $\{u_i\}$. After transforming to the standard random distributions, the stochastic response surface can be defined using the polynomial chaos expansion. The third-order Hermite polynomial chaos expansions can be written as

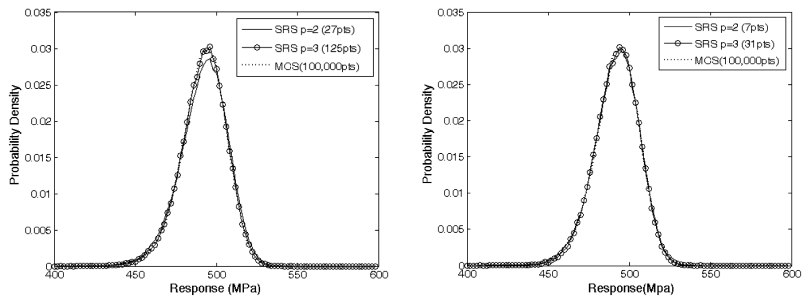
$$\begin{aligned} G^3 &= a_0^3 + \sum_{i=1}^{n} a_i^3 u_i + \sum_{i=1}^{n} a_{ii}^3 \left(u_i^2 - 1\right) \\ &+ \sum_{i=1}^{n} a_{iii}^3 \left(u_i^3 - 3u_i\right) + \sum_{i=1}^{n-1}\sum_{j>i}^{n} a_{ij}^3 u_i u_j \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ijj}^3 \left(u_i u_j^2 - u_i\right) + \sum_{i=1}^{n-2}\sum_{j>i}^{n-1}\sum_{k>j}^{n} a_{ijk}^3 u_i u_j u_k \end{aligned} \tag{14}$$

Note that the polynomials are constructed in the standard Gaussian space rather than the original design space. For the third-order expansion, the number of unknown coefficients are 20.

The coefficients of the polynomial chaos expansion are obtained using the model outputs at selected collocation points. The collocation points are selected from the roots of the polynomial that is one order higher than the polynomial chaos expansion (Villadsen and Michelsen, 1978). For example, to solve for a three-dimensional second-order polynomial chaos expansion, the roots of the third-order Hermite polynomial, $-\sqrt{3}$, 0 and $\sqrt{3}$ are used, thus the possible collocation points are (0, 0, 0), ($-\sqrt{3}$, $-\sqrt{3}$, $-\sqrt{3}$), ($-\sqrt{3}$, 0, $\sqrt{3}$), etc. There are 27 possible collocation points and 10 unknown coefficients in the case of second-order expansion. For robust estimation of the regression coefficients, the number of collocation points in general should be twice the number of unknown coefficients.

After choosing collocation points in the standard normal space, a transformation is applied from the standard Gaussian space to design space according to the PDF associated with the design variables. In the torque-arm model, the PDF of the performance function is plotted in Figure 3(a) for polynomials of different orders. The accuracy and the convergence of the SRS are compared with the PDF obtained using the Monte Carlo simulation with 100,000 sample points. As expected, the error is reduced for higher-order polynomials.

Figure 3 PDF of performance function $G(\mathbf{x})$ – torque-arm problem: (a) only function values are used and (b) function values and local sensitivities are used



(a) (b)

In order to reduce the required number of sampling points in constructing the SRS, local sensitivity information is also used. At each sampling point, n + 1 data are available (function value + gradients of n random variables). To account for the local sensitivity information, the expression in Equation (14) is differentiated with respect to the random variables. However, the SRS are defined in the standard Gaussian space. As a result, it is necessary to transform the local sensitivity in the design space into the standard Gaussian space using the following equation:

$$\nabla G(\mathbf{u}) = \nabla G(\mathbf{x}) \frac{\partial \mathbf{T}^{-1}(\mathbf{u})}{\partial \mathbf{u}} \tag{15}$$

where $\mathbf{T}$: $\mathbf{x} \to \mathbf{u}$ is the transformation between the design and standard Gaussian spaces.

Using local sensitivity, information increases by a factor of n the number of data obtained from each sampling point (from one to four for the case of three design variables), hence the number of sampling points can be reduced $n + 1$ times. In Figure 3(b), the PDFs of the performance function are plotted for alternative polynomial expansions and that obtained using the Monte Carlo simulation. Note that a stochastic response surface with the same level of accuracy to that showed in Figure 3(a) can be obtained with four times less number of sampling points.

6.2 Reliability-based design optimisation

The reliability optimisation problem under consideration requires to minimise the mass of the torque arm while satisfying stress reliability constraints. Let the model output G_i be defined as

$$G_i(\mathbf{x}) = 1 - \frac{\sigma_i}{\sigma_{\max}} \tag{16}$$

Using Equation (11), the design optimisation problem can be defined as

$$\begin{aligned} &\text{Minimise} && \text{Mass}(\mathbf{d}) \\ &\text{subject to} && P\left(G_i(\mathbf{x}) \le 0\right) \le \Phi\left(-\beta_t\right), \quad i = 1,\ldots,\text{NC} \\ &&& \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U \end{aligned} \tag{17}$$

where β_t is the target reliability index and Φ is the cumulative distribution function of the standard normal distribution. For the reliability analysis, a target reliability index of 3.0 is used, which is equivalent to 99.87% reliability. The stress values at four (i.e. NC = 4) different locations are monitored. Table 2 gives the lower and upper bounds of the mean values associated with the design variables (modelled as random variables). Since the design parameters are the relative movement of the corner points, the initial values for all design parameters is 0. The lower and upper bounds are chosen such that the topology of the boundary is preserved throughout the whole design process.

For comparison purposes, this RBDO problem is solved using all random variables without any adaptive reduction. At each design point, the eight random variables are used to construct the SRS. To generate the third-order SRS, a total of 89 sampling points are used; at each sampling point stress and local sensitivity information is gathered. The optimisation problem converges at the 21st iteration. The design variables at the optimum design are listed in the fourth column of Table 2, and the optimum geometry is plotted in Figure 4(a). Figure 4(b) shows the stress distribution of the torque-arm model at the optimum design. The maximum stress occurs at Point A with a value of 704 MPa. Considering the maximum allowable stress limit is 800 MPa, the mean value of the optimum design has about 96 MPa margin. Figure 5 shows the design history of the cost function. The initial mass of 0.878 kg is reduced to 0.522 kg (about 59.4%) at the optimum design. Most reduction has been achieved in the first five design cycles, and after that the optimisation slowly converged by adjusting design parameters.

Table 2 Definition of random design parameters and mean value bounds

Random variables	$\mathbf{d}^L$	$\mathbf{d}$ *(Initial)*	$\mathbf{d}^*$ *(Optimum)*	$\mathbf{d}^U$	*SD*	*Distribution type*
d_1	–3.0	0.0	–0.7532	1.0	0.1	Normal
d_2	–0.5	0.0	–0.5000	1.0	0.1	Normal
d_3	–1.0	0.0	–0.1346	1.0	0.1	Normal
d_4	–2.7	0.0	–2.5443	1.0	0.1	Normal
d_5	–5.5	0.0	–0.8508	1.0	0.1	Normal
d_6	–0.5	0.0	1.9998	2.0	0.1	Normal
d_7	–1.0	0.0	0.8319	7.0	0.1	Normal
d_8	–0.5	0.0	0.0000	0.0	0.1	Normal

Figure 4 Optimum design and stress distribution of the torque-arm model with eight random variables: (a) --- – initial design, — optimum design and (b) maximum equivalent stress = 704 MPa at point A

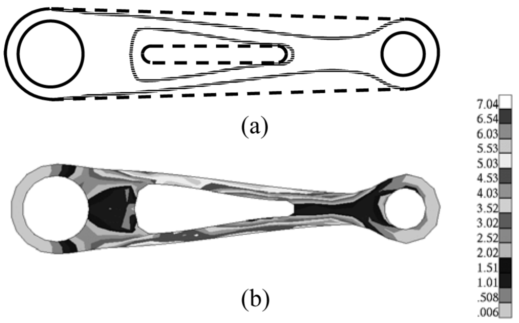
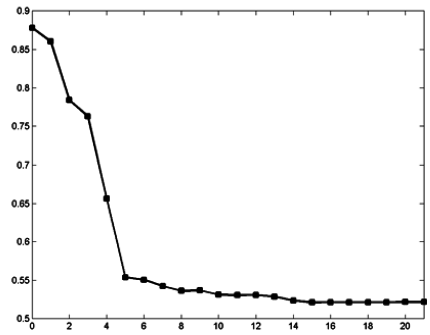


Figure 5 Optimisation history of cost function (mass) for the torque-arm model with eight random variables



6.3 Reduction of random variables

The RBDO problem in the previous section was solved with all random variables. However, some random variables did not significantly contribute to the stress function variance. Thus, a significant amount of computational cost can be saved if the random variables whose contribution to the variance of the output is small are considered as deterministic variables at their mean values. This section describes how the GSI considering only main factors that can be used for deciding whether to fix unessential random variables during the construction of SRS.

At the initial design stage, a low-order stochastic response surface is constructed using all random variables. In this particular example, the first-order SRS is constructed using 17 sampling points. At the initial design, the first-order SRS with eight random variables can be expressed as,

$$\begin{aligned} G^1 &= a_0 + a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 + a_5u_5 + a_6u_6 + a_7u_7 + a_8u_8 \\ &= 4.95 + 0.0063u_1 + 0.117u_2 + 0.00008u_3 - 0.0019u_4 \\ &\quad +0.0026u_5 - 0.052u_6 - 0.0002u_7 - 0.016u_8 \end{aligned} \tag{18}$$

One useful aspect of the polynomial chaos expansion is that the coefficients in Equation (18) are a measure of the contribution of the corresponding random variable to the variation of the output, and these coefficients will not change significantly in higher-order SRS. In addition, typically the global sensitivity index associated with a particular variable is responsible for most of its contribution to the output variance. Thus, evaluating the GSI using the first-order SRS can be justified. As all random variables are transformed into standard normal random variables, the variance of G^1 can be evaluated analytically. Using Equation (10), the global sensitivity index of each random variable is calculated. Using Eqution (18) and assuming the design variables are independent, the global sensitivity index can be calculated as:

$$S_i = \frac{a_i^2}{\sum_{j=1}^{n} a_j^2} \tag{19}$$

Note that the denominator in Equation (19) is the total variance of G^1 using the first-order approximation. Thus, the global sensitivity index, S_i, is the ratio of the contribution of *i*th random variable to the total variance. If the global sensitivity index of a specific variable is less than a threshold value, the variable is considered as deterministic and fixed at its mean value.

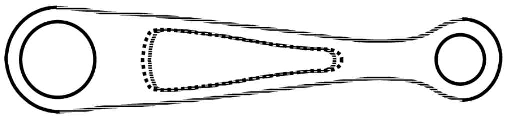
To show the advantage of fixing unessential random variables, the GSI of the torque-arm model are calculated. Table 3 gives the GSI of the torque-arm model using the first-order SRS at the initial design. The total variance of stress function is 1.670×10^{-2}. On the basis of the GSI, there are only three random variables whose contribution is greater than 1.0%; that is, u_2, u_6 and u_8. Thus, in the reliability analysis, only these three random variables are used in constructing the third-order SRS, which now requires only 19 sampling points. All other random variables are considered as deterministic variables at their mean values. If the total number of sampling points for both low (17) and higher-order (19) polynomial expansions are compared with the higher-order SRS using all random variables (89), a significant reduction of the number of sampling points was achieved.

The RBDO problem, defined in Equation (17) is now solved using the proposed reduction of random variables. The optimisation algorithm converges after the 17th iteration. As seen in Figure 6, the optimum design using the adaptively reduced SRS is slightly different from that obtained in the previous section (without adaptive reduction). The former has a longer interior cutout than the latter. This can be explained from the fact that some variables were considered deterministic throughout the design process. Furthermore, the optimum value achieved using the adaptively reduced SRS converges to a lower value than the one (without adaptive reduction). The total mass of the torque arm is reduced in 57.6%. The difference between the two approaches is approximately 1.8%.

Table 3 Global sensitivity indices considering only main factors for the torque-arm model at the initial design. Only three random variables (u_2, u_6 and u_8) are preserved when a threshold value of 1.0% is in place

SRV	*Variance*	*GSI (%)*
u_1	3.916×10^{-5}	0.235
u_2	1.369×10^{-2}	*82.0*
u_3	6.403×10^{-9}	0.00003834
u_4	3.667×10^{-6}	0.02197
u_5	6.864×10^{-6}	0.04109
u_6	2.702×10^{-3}	*16.179*
u_7	4.818×10^{-8}	0.0002885
u_8	2.538×10^{-4}	*1.519*

Figure 6 Optimum designs for the full SRS (solid line) and adaptively reduced SRS (dotted line). Because some variables are fixed, the interior cutout of the design from the adaptively reduced SRS is larger than that from the full SRS



The number of active random variables associated with the modelling of the first constraint during the design iterations are listed in Table 4. On average, four random variables were preserved as such, which implies that only 29 sampling points were required for constructing the SRS. This is three times less than the SRS approach without adaptive reduction (89 sampling points).

Table 4 Comparison of the number of random variables in each design cycle. The threshold of 1.0% is used. The first constraint is listed

Design cycle	*Full SRS*	*Reduced SRS*
1	8	3
2	8	3
3	8	3
4	8	3
5	8	4
6	8	4
7	8	5
8	8	4
⋮	⋮	⋮
17	8	4
⋮	⋮	–

7 Conclusions

In this paper, we present an approach for solving reliability-based optimisation problems involving a computationally demanding model. Key aspects of the approach are:

1. the uncertainty propagation of random variables using a polynomial chaos expansion and local sensitivity information and
2. the use of global sensitivity information to adaptively reduced the number of random variables throughout the design process.

The convergence and accuracy of the proposed approach was demonstrated using a benchmark case and an industrial reliability-based optimisation problem (automotive part).

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Nomenclature

$\mathbf{u}$	vector of standard normal random variables
$\mathbf{x}$	vector of random variables
$\mathbf{d}$	vector of design parameters
$\Gamma_p(u_i,\ldots,u_p)$	multidimensional Hermite polynomials of degree p
$[\mathbf{K}]$	structural stiffness matrix
$\{\mathbf{F}\}$	structural load vector
$\{\mathbf{D}\}$	nodal solution vector (displacement)
S_i	global sensitivity index of ith random variable
S_i^{total}	total sensitivity index of ith random variable
$\mathrm{COV}(\cdot,\cdot)$	covariance of two random variables
$E(\cdot)$	expected value
$V(\cdot)$	variance
G^p	pth-order stochastic response approximation
$\sigma_{\max}$	maximum allowable equivalent stress
$c(\mathbf{d})$	cost function
P_f	failure probability
$p(\mathbf{x})$	joint PDF of input random variable $\mathbf{x}$
β_t	target reliability index
Φ	cumulative distribution function of the standard normal random variable