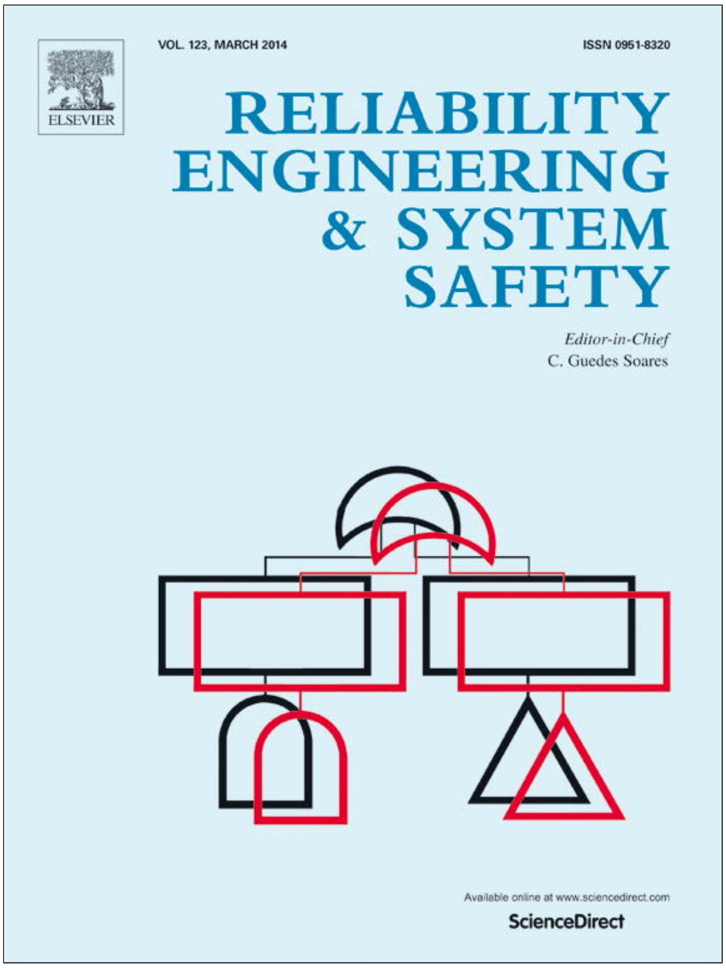
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How coupon and element tests reduce conservativeness in element failure prediction

Chan Y. Park, Nam H. Kim*, Raphael T. Haftka

Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, FL 32611, USA

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ABSTRACT

Structural elements, such as stiffened panels, are designed by combining material strength data obtained from coupon tests with a failure theory for 3D stress field. Material variability is captured by dozens of coupon tests, but there remains epistemic uncertainty due to error in the failure theory, which can be reduced by element tests. Conservativeness to compensate for the uncertainty in failure prediction (as in the A- or B-basis allowables) results in a weight penalty. A key question, addressed here, is what weight penalty is associated with this conservativeness and how much it can be reduced by using coupon and element tests. In this paper, a probabilistic approach is used to estimate the conservative element failure strength by quantifying uncertainty in the element strength prediction. A convolution integral is used to efficiently combine uncertainty from coupon tests and that from the failure theory. Bayesian inference is then employed to reduce the epistemic uncertainty using element test results. The methodology is examined with typical values of material variability (7%), element test variability (3%), and the error in the failure theory (5%). It is found that the weight penalty associated with no element test is significant (20% heavier than an infinite number of element tests), and it is greatly reduced by more element tests (4.5% for 5 element tests), but the effect of the number of coupon tests is much smaller.

1. Introduction

Uncertainty has always been a major concern in structural design. For example, predicting the strength of a structural element has two major sources of epistemic uncertainty (uncertainty associated with the lack of information). The first comes from errors in failure prediction based on calculated stresses and a failure theory. The second source is errors in measuring variability of material properties. Coupon tests are performed to measure material variability, but the estimated variability has error due to the limited number of coupons.

Aircraft designers use conservative measures, such as A- or B-basis allowable, to compensate for uncertainty in material strength prediction as in MIL-HDBK [1]. For example, the B-basis introduces conservativeness in two ways. To compensate for variability, the B-basis uses the lower 10% value of the material strength distribution. However, calculating the lower 10% relies on the number of coupons, which brings in epistemic uncertainty. Thus, the B-basis requires an additional 95% confidence level to compensate for the epistemic uncertainty. That is, the B-basis provides a value that belongs to the lower 10% with 95% probability. The B-basis is calculated based on a sample mean and standard deviation with a factor for one-sided tolerance limit with an assumed population distribution. MIL-HDBK [1] and Owen et al. [2] presented tables of the factors with various population distributions. To compensate for the error in a failure theory, it is common practice to repeat element tests three times and then select the lowest test result as a conservative estimate of the failure envelope; this process can be interpreted as applying a knockdown factor on the average test result.

Treating epistemic uncertainty is reflected in the literature of probabilistic design. Noh et al. [3] compensated for epistemic uncertainty caused by the finite number of samples with a confidence level of 97.5%. Matsumura et al. [4] and Villanueva et al. [5] considered the effect of epistemic uncertainty in a computer model on estimating probability of failure of an integrated thermal protection system of a space vehicle and demanded 95% confidence for the epistemic uncertainty.

These conservative statistical approaches have worked successfully to achieve the safety of structural designs. However, they were applied at an individual test stage without considering their overall efficiency to achieve the safety level at the final stage. Also, it has not been quantified how much these tests reduce the weight penalty compared to the design without tests.

When we use failure theory to predict the strength of an element, we propagate uncertainty in coupons and combine it with uncertainty in the failure theory. We build and test the

* Corresponding author. Tel.: +1 352 575 0665; fax: +1 352 392 7303.
E-mail addresses: cy.park@ufl.edu (C.Y. Park), nkim@ufl.edu (N.H. Kim), haftka@ufl.edu (R.T. Haftka).

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Nomenclature

b_e	error bound for failure theory
b_σ	estimated bound for standard deviation of structural element
$\hat{e}_{k,Ptrue}$	possible true error in failure theory
$f^{init}(\mu_{e,Ptrue}, \sigma_{e,Ptrue})$	initial joint PDF for given mean and standard deviation of structural element
$f_{k,Ptrue}(e_{k,Ptrue})$	PDF for given possible true error in failure theory
$f_{\mu c,Ptrue}(\mu_{c,Ptrue})$	PDF for given possible true mean of material strength
$f_{\mu e,Ptrue}(\mu_{e,Ptrue})$	PDF for given possible true mean of structural strength
$f_{\sigma c,Ptrue}(\sigma_{c,Ptrue})$	PDF for given possible true standard deviation of material strength
$f_{\sigma e,Ptrue}(\sigma_{e,Ptrue})$	PDF for given possible true standard deviation of structural strength
$f^{upd}(\mu_{e,Ptrue}, \sigma_{e,Ptrue})$	updated joint PDF for given mean and standard deviation of structural element
$f^{upd}_{\mu e,Ptrue}(\mu_{e,Ptrue})$	updated marginal distribution for given mean of structural element
$f^{upd}_{\sigma e,Ptrue}(\sigma_{e,Ptrue})$	updated marginal distribution for given standard deviation of structural element
$k_{3d,calc}$	calculated ratio of structural element strength to material strength
$\hat{k}_{3d,Ptrue}$	possible true structural element strength to material strength
$k_{3d,true}$	true ratio of structural element strength to material strength
$l^i_{test}(\mu_{e,Ptrue}, \sigma_{e,Ptrue})$	likelihood function of ith test for given mean and standard deviation of structural element
$\mu_{0.05}$	mean of 5th percentile of the mean element strength for given test result
$\hat{\mu}_{c,Ptrue}$	possible true mean of material strength
$\mu_{c,test}$	measured mean of material strength from coupon test
$\mu_{c,true}$	true mean of material strength
$\hat{\mu}_{e,Ptrue}$	possible true mean of structural element strength
$\mu_{e,test}$	measured mean of structural element strength from coupon test
$\mu_{e,true}$	true mean of structural element strength
n_c	the number of coupon tests
n_e	the number of element tests
PUD	probability of unconservative design
PTD	possible true distribution
$\hat{\sigma}_{c,Ptrue}$	possible true standard deviation of material strength
$\sigma_{c,test}$	measured standard deviation of material strength from coupon test
$\sigma_{c,true}$	true standard deviation of material strength
$\hat{\sigma}_{e,Ptrue}$	possible true standard deviation of structural element strength
$\sigma_{e,test}$	measured standard deviation of structural element strength from coupon test
$\sigma_{e,true}$	true standard deviation of structural element strength
$\tau_{0.05}$	5th percentile of the mean element strength for given test results
$\hat{\tau}_{c,Ptrue}$	possible true material strength
$\hat{\tau}_{c,true}$	true material strength
$\hat{\tau}_{e,Ptrue}$	possible true structural element strength
$\hat{\tau}_{e,true}$	true structural element strength
$w_{0.95}$	95th percentile of the weight penalty for given test results

Superscripts

init	initial distribution (prior distribution)
upd	updated distribution (posterior distribution)

Subscripts

calc	calculated value using a theory
Ptrue	possible true estimate reflecting epistemic uncertainty of estimation process
test	measured value from a test
true	true value

structural element in order to reduce the combined uncertainty. The remaining uncertainty after tests depends on the numbers of coupons and elements. Coupon tests are relatively cheap compared to element tests, and therefore, we usually perform many more coupon tests (several dozens) than element tests (a handful). The objective of this paper is to model the effect of these tests on making a conservative element strength prediction with a 95% confidence level by quantifying the uncertainty in the prediction process and to analyze the tradeoff between the number of coupons and elements for reducing the conservativeness.

In the overview of future structure technology for military aircraft, Joseph et al. [6] noted that a progressive uncertainty reduction model, which is seen in building-block tests, can be a feasible solution today, since a complete replacement of traditional tests with computational models is not feasible yet. Lincoln et al. [7] pointed out that building-block tests play a key role in reducing errors in failure prediction of composite structures due to large uncertainty in computational models. They noted that the use of probabilistic methods can significantly lower the test cost by reducing the scope of the test program.

There are also several studies investigating the effect of tests on safety and reducing uncertainty in computational models. Jiao and Moan [8] investigated the effect of proof tests on structural safety using Bayesian inference. They showed that proof tests reduce uncertainty in the strength of a structure, and thus provide a substantial reduction in the probability of failure. An et al. [9] investigated the effect of structural element tests on reducing uncertainty in element strength using Bayesian inference. Acar et al. [10] modeled a simplified building-block process with safety factors and knockdown factors. Bayesian inference is used to model the effect of structural element tests. They show the effect of the number of tests on the design weight for the same probability of failure, and vice versa. Jiang and Mahadevan [11] studied the effect of tests in validating a computational model by obtaining an expected risk in terms of the decision cost. Urbina and Mahadevan [12] assessed the effects of system level tests for assessing reliability of complex systems. They built computational models of a system and predicted the performance of the system. Tests are then incorporated into the models to estimate the confidence in the performance of the systems. Park et al. [13] estimated uncertainty in computational models and developed a methodology to evaluate likelihood using both test data and a computational model. McFarland and Bichon [14] estimated probability of failure by incorporating test data for a bistable MEMS device.

In this paper, we assume that with an infinite number of coupons and elements, the epistemic uncertainty associated with samples and failure theory can be eliminated. With a finite number of tests, the epistemic uncertainty is compensated for by using a conservative mean value at the 95% confidence level, in the context of the B-basis. The aleatory uncertainty can then be compensated for either by 90% of the population (deterministic design) or by specifying probabilities of failure. We focus on the effect of the number of tests on the conservative estimate of element strength and the resulting weight penalty compared to the case with an infinite number of tests. To have the conservative estimate, we predict the mean element strength and its uncertainty by combining two uncertainties from coupon tests and a failure theory, using a convolution integral. Then, Bayesian inference is incorporated with element tests in order to reduce the uncertainty. With the proposed two-stage uncertainty model, it is possible to identify the effect of two types of tests on reducing uncertainty and corresponding conservativeness and weight penalty.

The paper is organized as follows: Section 2 introduces the building-block test process used in this paper, which is composed of coupon and element test stages, and sources of uncertainty. Section 3 provides uncertainty modeling of the building-block test process to estimate the element strength and its uncertainty. This section has three subsections: coupon tests, element design and element tests. Section 4 introduces different measures that are used to evaluate the efficiency of different tests. Section 5 presents numerical results, followed by conclusions in Section 6.

2. Structural uncertainties

For aircraft structures, the building-block test process (Fig. 1) is used to find design errors and to reduce uncertainties in design and manufacturing. At each level, analytical/numerical models are calibrated to account for discrepancies between model prediction and test results. Since the errors are unknown at the modeling stage, they may be modeled as uncertainty (epistemic), and test results may be used to reduce the uncertainty. Starting from simple coupon tests at the bottom level, structural complexity gradually increases further up the building-block pyramid. The number of tests gradually reduces from bottom to top; for example, 50 coupons, 3 elements, and 1 component. In higher-level tests, it is difficult to understand deviations from analytical predictions, tests are more expensive and any design modification can be expensive. The building-block test process is designed to detect modeling errors at a lowest level.

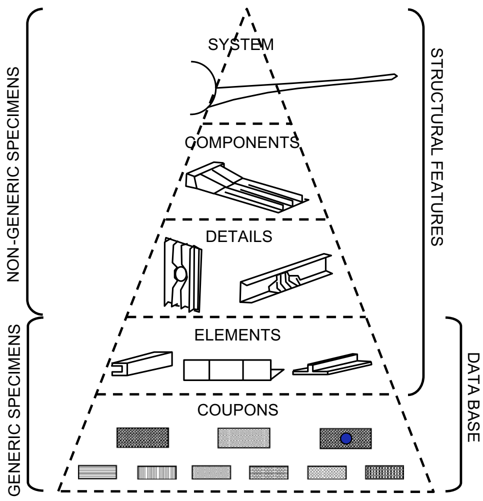

Fig. 1. Building-block test process for aircraft structural components.

Although building-block tests are designed to reduce uncertainty, it is difficult to quantify how much tests in each level can contribute to uncertainty reduction, which is the main objective of this paper. Once the contribution of tests to uncertainty reduction is understood, a design engineer can decide how to allocate resources to different levels in order to achieve the target reliability at minimum cost.

Although the actual building-block test process has many levels, this paper only considers coupon and element tests to demonstrate the effect of these tests on uncertainty reduction. Table 1 shows the objectives of these two tests and the sources of uncertainty.

In this paper, the failure stress of a structural element is simulated with randomly generated test results. True distributions are used only for generating test samples and assessing the estimated failure stress.

3. Modeling uncertainty in the building-block test process

In order to model the two-level building-block test process, it is assumed that the strength of coupons and elements follows a normal distribution due to material variability. This assumption can easily be removed when actual test results are available and the type of distribution can be identified using various statistical methods, such as the one in MIL-HDBK [1]. In the following subsections, uncertainties at each stage are modeled.

3.1. Coupon tests: modeling uncertainty in estimating statistical properties

Due to inherent variability, the material strength shows a statistical distribution. Coupon tests are conducted to estimate the distribution and to determine regulatory (e.g., FAA) strength allowables (e.g., A-basis or B-basis) that compensate for the uncertainty. It is assumed that the true material strength, $\hat{\tau}_{c,true}$, follows a normal distribution, as

$$\hat{\tau}_{c,true} \sim N(\mu_{c,true}, \sigma_{c,true}) \quad (1)$$

where $\mu_{c,true}$ and $\sigma_{c,true}$ are, respectively, the mean and standard deviation of $\hat{\tau}_{c,true}$. The circumflex symbol represents a random variable. The subscript "c" is used to denote coupons. In this paper, Eq. (1) is only used for the purpose of simulating coupon tests; the true distribution is unknown to the designer.

Since the true distribution parameters are estimated with a finite number of coupons, the estimated parameters have sampling uncertainty (or error). Thus, it is natural to consider these parameters as distributions rather than deterministic values. In this paper, this estimated distribution is called the possible true distribution (PTD) of the parameter. For example, if $\mu_{c,true}$ is estimated from 50 coupons, with a sample mean of 1.02 and sample standard deviation of 0.1, then the PTD of the mean is a distribution following $N(1.02, 0.1)$.

Table 1
Sources of uncertainty in the building-block test process for estimating element strength.

Test stage	Objectives	Uncertainty sources
Coupon test	Estimate mean value and variability of material strength	Sampling error due to a finite number of coupons
Element design	Estimate multi-axial strength based on a failure theory	Incomplete knowledge of failure mechanism: error in failure theory
Element test	Reduce uncertainty in the multi-axial strength	Sampling error due to a finite number of elements

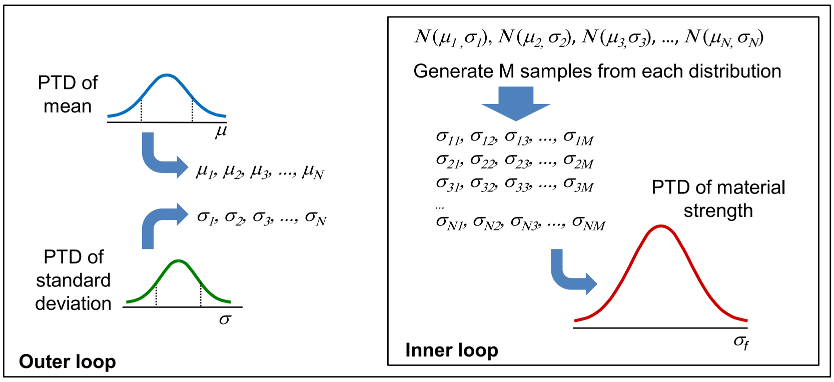

Fig. 2. Double-loop Monte Carlo simulation to obtain the possible true distribution of material strength.

In this setting, the estimated material strength essentially becomes a distribution of distributions. The PTD of material strength can be obtained using a double-loop Monte Carlo simulation (MCS), as shown in Fig. 2. In figure, the outer loop generates N samples of the two distribution parameters, from which N pairs of normal distributions, $N(\mu_i, \sigma_i)$, can be defined. In the inner loop, M samples of material strengths are generated from each $N(\mu_i, \sigma_i)$. Then, all $N \times M$ samples are used to obtain the PTD of material strength, which includes both material variability and sampling errors.

In order to model the above MCS process analytically, the PTD of material strength, $\hat{\tau}_{c,Ptrue}$, is firstly defined as a conditional distribution as

$$\hat{\tau}_{c,Ptrue} | (\hat{\mu}_{c,Ptrue} = \mu_{c,Ptrue}, \hat{\sigma}_{c,Ptrue} = \sigma_{c,Ptrue}) \sim N(\mu_{c,Ptrue}, \sigma_{c,Ptrue}) \quad (2)$$

where the left-hand side is a conditional random variable given $\mu_{c,Ptrue}$ and $\sigma_{c,Ptrue}$. Since $\hat{\mu}_{c,Ptrue}$ and $\hat{\sigma}_{c,Ptrue}$ are random, Eq. (2) corresponds to an incident of possible true distributions. In Fig. 2, randomly generated μ_i and σ_i correspond to $\mu_{c,Ptrue}$ and $\sigma_{c,Ptrue}$, respectively.

Note that $\hat{\mu}_{c,Ptrue}$ and $\hat{\sigma}_{c,Ptrue}$ depend on the number of coupons. With n_c coupons, $\hat{\mu}_{c,Ptrue}$ is nothing but the distribution of the sample mean and can be estimated as

$$\hat{\mu}_{c,Ptrue} \sim N\left(\mu_{c,test}, \frac{\sigma_{c,test}}{\sqrt{n_c}}\right) \quad (3)$$

where $\mu_{c,test}$ and $\sigma_{c,test}$ are, respectively, the mean and standard deviation of coupons. With an infinite number of coupons, $\hat{\mu}_{c,Ptrue}$ will become a deterministic value; i.e., no sampling error.

It is also well-known that the standard deviation $\hat{\sigma}_{c,Ptrue}$ follows a chi-distribution of order $n_c - 1$. In a way similar to the mean, $\hat{\sigma}_{c,Ptrue}$ can be estimated as

$$\hat{\sigma}_{c,Ptrue} \sim \frac{\sigma_{c,test}}{\sqrt{n_c - 1}} \chi(n_c - 1) \quad (4)$$

where $\chi(n_c - 1)$ is the chi-distribution of the order $n_c - 1$ [15].

Let $f_{\mu c,Ptrue}(\mu_{c,Ptrue})$ and $f_{\sigma c,Ptrue}(\sigma_{c,Ptrue})$ be the PDFs of $\hat{\mu}_{c,Ptrue}$ and $\hat{\sigma}_{c,Ptrue}$, respectively. Then, the PDF of $\hat{\tau}_{c,Ptrue}$ is derived as

$$f_{c,Ptrue}(\tau_{c,Ptrue}) = \int_0^{\infty} \int_{-\infty}^{\infty} \varphi(\tau_{c,Ptrue} | \mu_{c,Ptrue}, \sigma_{c,Ptrue}) f_{\mu c,Ptrue}(\mu_{c,Ptrue}) f_{\sigma c,Ptrue}(\sigma_{c,Ptrue}) \, d\mu_{c,Ptrue} \, d\sigma_{c,Ptrue} \quad (5)$$

where the notation $\varphi(x|a, b)$ denotes the value of a normal PDF with mean a and standard deviation b at x.

Fig. 3 compares the PDF of $\hat{\tau}_{c,true} \sim N(1.1, 0.077)$ with that of $\hat{\tau}_{c,Ptrue}$ with different numbers of coupons. In the case of 30 coupons, the samples have $\mu_{c,test} = 1.053$ and $\sigma_{c,test} = 0.096$. Using Eqs. (3) and (4), the standard deviations of $\hat{\mu}_{c,Ptrue}$ and $\hat{\sigma}_{c,Ptrue}$ are estimated to be 0.018 and 0.013, respectively, which reflect the randomness of the samples. Note that in the case of 30 coupons, the mean was slightly underestimated, but a large standard deviation compensates for it. In the case of 80 coupons, the samples have $\mu_{c,test} = 1.113$ and $\sigma_{c,test} = 0.083$. The standard deviations of $\hat{\mu}_{c,Ptrue}$ and $\hat{\sigma}_{c,Ptrue}$ are 0.009 and 0.007, respectively. As expected, $\hat{\tau}_{c,Ptrue}$ with 80 coupons yields a narrower estimate than that of 30 coupons.

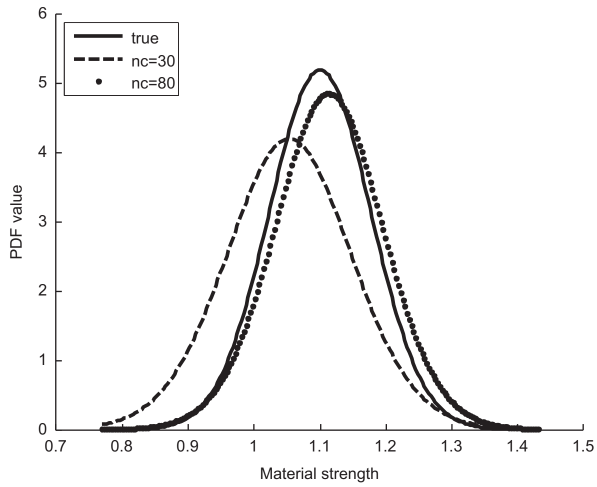

Fig. 3. The distribution of possible true material strength estimated with 30 and 80 coupons, sampled from the true distribution $\hat{\tau}_{c,true} \sim N(1.1, 0.077)$ (solid curve).

3.2. Element design: combining uncertainties

To design a structural element, the material strength from coupon tests must be generalized to multi-axial stress states using a failure theory. Since the failure theory is not perfect, additional error (i.e., epistemic uncertainty) is introduced, which needs to be combined with the sampling error in the coupon test. Since the uncertainty in element strength can be represented using the distributions of the mean and standard deviation, the uncertainties of these two random variables are modeled separately [16].

A failure theory provides a relation between uni-axial strength and multi-axial strength. In this paper, this relation is represented using a prediction factor $k_{3d,true}$ as

$$\tau_{e,true} = k_{3d,true} \tau_{c,true} \quad (6)$$

where $\tau_{c,true}$ is a true uni-axial material strength, and $\tau_{e,true}$ is a true multi-axial equivalent strength. Subscript "e" is used to denote that the variable is for an element. For example, when the von Mises criterion is used, $k_{3d,true} = 1$. The relation between the two

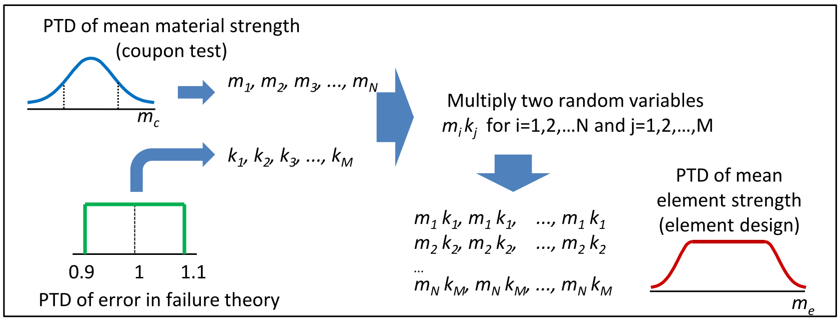

Fig. 4. Process of estimating element mean strength.

mean values can be obtained from Eq. (6) as

$$\mu_{e,true} = k_{3d,true}\mu_{c,true} \tag{7}$$

Again, $k_{3d,true}$ is unknown to designers; only its estimate $k_{3d,calc}$ is given from the failure theory. Therefore, the epistemic uncertainty in the failure theory can be represented using the PTD of the prediction factor as

$$\hat{k}_{3d,Ptrue} = (1-\hat{e}_{k,Ptrue})k_{3d,calc} \tag{8}$$

where error $\hat{e}_{k,Ptrue}$ is assumed to follow a uniform distribution with bounds of $\pm b_e$, which reflect the designer's confidence in the failure theory. Then, the designer's estimated relationship corresponding to Eq. (7) can be written as

$$\hat{\mu}_{e,Ptrue} = \hat{k}_{3d,Ptrue}\hat{\mu}_{c,Ptrue} \tag{9}$$

Fig. 4 shows the process of obtaining $\hat{\mu}_{e,Ptrue}$ through MCS. First, N samples from $\hat{\mu}_{c,Ptrue}$ and M samples from $\hat{k}_{3d,Ptrue}$ are generated. Then, $\hat{\mu}_{e,Ptrue}$ is estimated from $N \times M$ samples that are obtained by taking every possible combination of the two sets of samples.

In this paper, a convolution integral is used to calculate the PDF of $\hat{\mu}_{e,Ptrue}$. The convolution integral provides an accurate PDF using numerical integration, whereas MCS brings in additional uncertainty. A comparison between MCS and the convolution integral is given in the example section. In the case of a normally distributed mean and uniformly distributed error, the PDF of $\hat{\mu}_{e,Ptrue}$ can be written as

$$f_{\mu e,Ptrue}(\mu_{e,Ptrue}) = \int_{\frac{\mu_{e,Ptrue}}{(1-b_e)}}^{\frac{\mu_{e,Ptrue}}{(1+b_e)}} \frac{1}{2b_e\mu_{c,Ptrue}}\varphi\left(\mu_{c,Ptrue}|\mu_{c,test}, \frac{\sigma_{c,test}}{\sqrt{n_c}}\right) d\mu_{c,Ptrue} \tag{10}$$

where b_e is the error bound of $\hat{e}_{k,Ptrue}$ and $k_{3d,calc}=1.0$ is assumed. See Appendix A for detailed derivations. The integral domain is divided to 200 segments, and the integral is evaluated using Gaussian quadrature with 3 points for each of the 200 segments.

Fig. 5 shows the PDF of typical $\hat{\mu}_{e,Ptrue}$ for $n_c=10$ and 50. As the number of coupons increases, the PDF approaches a uniform distribution, which corresponds to the uncertainty in the failure theory. When the number of coupons is small, the distribution has a long tail because of sampling errors in the coupon tests. This $\hat{\mu}_{e,Ptrue}$ serves as the prior distribution representing the designer's knowledge before element tests.

Unlike the mean, there is only a weak relationship between the standard deviation of coupon strength and that of element strength. Usually test conditions are well controlled to minimize uncertainty; the standard deviation of the test is substantially smaller than that of material properties. The distribution of $\hat{\sigma}_{e,Ptrue}$ is defined as a uniform distribution with lower and upper bounds as

$$f_{\sigma e,Ptrue}(\sigma_{e,Ptrue}) = \frac{1}{(\sigma_e^{upper}-\sigma_e^{lower})} I(\sigma_{e,Ptrue} \in [\sigma_e^{lower}, \sigma_e^{upper}]) \tag{11}$$

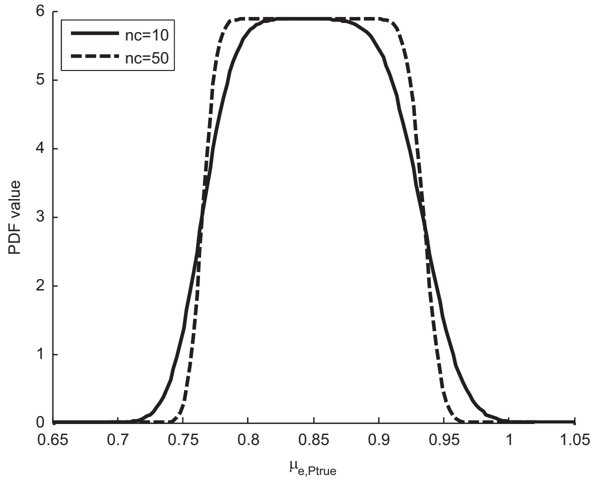

Fig. 5. Distributions of estimated element mean strength for different numbers of coupons ($b_e=0.1$, $\mu_{c,true}=0.85$, $\sigma_{c,true}=0.068$, and $k_{3d,calc}=1.0$).

where $I(\bullet)$ is the indicator function, and σ_e^{upper} and σ_e^{lower} are upper and lower bounds of the standard deviation of element strength, respectively. These bounds are estimated to cover a true standard deviation of the element test.

3.3. Element tests: Bayesian inference to reduce errors

The PTDs in Eqs. (10) and (11) are the combined uncertainties from (a) material variability, (b) sampling errors in coupon tests and (c) error in the failure theory. Although material variability will always exist, the other two epistemic uncertainties can be reduced using element tests. In this section, the effect of element tests on reducing uncertainty is modeled using Bayesian inference.

For the purpose of Bayesian inference, Eqs. (10) and (11) are used as marginal prior distributions. Since no correlation information is available, these distributions are assumed to be independent. Therefore, the prior joint PDF is given as

$$f^{init}(\mu_{e,Ptrue}, \sigma_{e,Ptrue}) = f_{\mu e,Ptrue}(\mu_{e,Ptrue}) f_{\sigma e,Ptrue}(\sigma_{e,Ptrue}) \tag{12}$$

In Bayesian inference, the updated joint PDF with n_e number of element tests is expressed as

$$f^{upd}(\mu_{e,Ptrue}, \sigma_{e,Ptrue}) = \frac{1}{A}\prod_{i=1}^{n_e} \ell_{test}^{i}(\mu_{e,Ptrue}, \sigma_{e,Ptrue}) f^{init}(\mu_{e,Ptrue}, \sigma_{e,Ptrue}) \tag{13}$$

where A is a normalizing constant and $\ell_{test}^{i}(\mu_{e,Ptrue}, \sigma_{e,Ptrue})$ is the ith likelihood function for given $\mu_{e,Ptrue}$, $\sigma_{e,Ptrue}$. From the assumption that the true element strength $\hat{\varepsilon}_{e,true}$ follows a normal distribution and by ignoring errors associated with the test, the likelihood function can be defined as a probability of obtaining test result

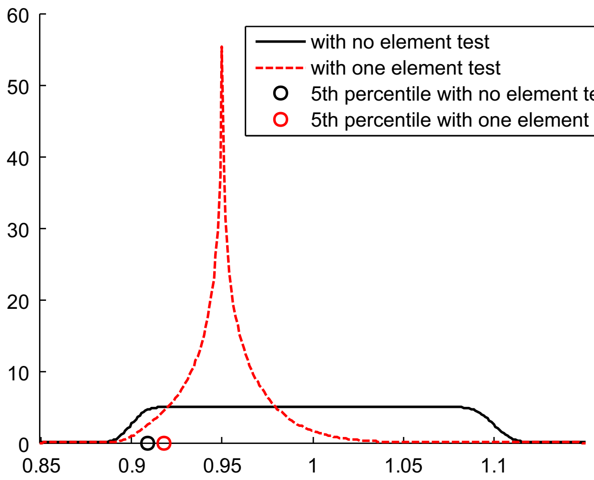

Fig. 6. Comparison of uncertainty in mean element strength before and after the first element test. The distribution with solid line before test is obtained with 50 coupon tests that happen to have the correct mean (1.0). The true element strength is 0.95, and the updated distribution with dashed line is given for an element test that has no error.

$\tau^i_{e,test}$ for given $\mu_{e,Ptrue}$ and $\sigma_{e,Ptrue}$ as

$$\ell^i_{test}(\mu_{e,Ptrue}, \sigma_{e,Ptrue}) = \varphi(\tau^i_{e,test} | \mu_{e,Ptrue}, \sigma_{e,Ptrue}) \quad (14)$$

Note that the likelihood function is not a probability distribution, but a conditional probability. The numerical scheme to evaluate the updated joint PDF is explained in Appendix B.

Using the updated joint PDF, the marginal PDFs of $\mu_{e,Ptrue}$ and $\sigma_{e,Ptrue}$ can be obtained as

$$f^{upd}_{\mu e,Ptrue}(\mu_{e,Ptrue}) = \int_0^{\infty} f^{upd}(\mu_{e,Ptrue}, \sigma_{e,Ptrue})\, d\sigma_{e,Ptrue} \quad (15)$$

$$f^{upd}_{\sigma e,Ptrue}(\sigma_{e,Ptrue}) = \int_{-\infty}^{\infty} f^{upd}(\mu_{e,Ptrue}, \sigma_{e,Ptrue})\, d\mu_{e,Ptrue} \quad (16)$$

The above distributions represent the uncertainty in estimating the mean and standard deviation of the element strength. The standard deviations of distributions in Eqs. (15) and (16) are measures of remaining uncertainty after the element tests.

3.4. Conservative prediction based on the updated possible true distribution

In common practice, element strength is calculated based on the lowest element test result of three tests and used as a design allowable. The process can be interpreted as applying a knock-down factor on the test average. In this paper, a conservative estimate of the mean element strength is used as a design allowable.

If a conservative estimate is wanted, the low 5th percentile of $f^{upd}_{\mu e,Ptrue}$ can be used for the 95% confidence level. The mean values of distributions in Eqs. (15) and (16) are, respectively, the estimate of mean and standard deviation of element strength. The 5th percentile of the marginal PDF for the mean element strength is expected to be less than the true mean element strength with a 95% confidence level. The 5th percentile, $\tau_{0.05}$, is calculated using Eq. (15) as

$$\int_{-\infty}^{\tau_{0.05}} f^{upd}_{\mu e,Ptrue}(x)\, dx = 0.05 \quad (17)$$

Fig. 6 illustrates the effect of one element test on calculating the 5th percentile. In this illustration, we choose the coupon tests to have the true mean, and the element test to have the true mean element strength. Since the true means of coupon and element tests are assumed to 1 and 0.95, respectively, the element strength prediction based on failure theory is unconservative by 5%. However, by taking, the 5th percentile, the error in element strength prediction is compensated. The figure shows the substantial reduction in uncertainty afforded even by a single element test. As a consequence, the 5th percentile (black and red circles) is actually higher after the test, allowing a reduction in the weight.

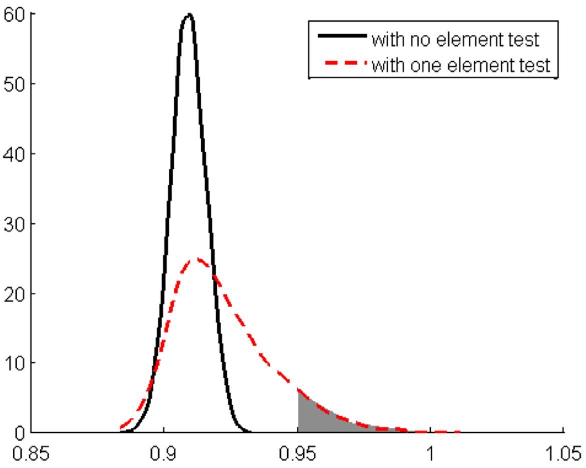

Fig. 7. Distributions of 5th percentiles with no element and with one element tests while the true mean element strength is 0.95.

The results shown in Fig. 6 are merely an illustration for a particular set of coupon and element test results. To see a general observation, we repeat evaluating the 5th percentile for randomly selected N sets of coupon and element test results (N=100,000 here), from which the distributions of 5th percentiles shown in Fig. 7 are obtained. Due to variability in test, the 5th percentiles also have variability, which are shown as distributions in Fig. 7.

In Fig. 7, the area of the gray shade is the probability of having the 5th percentile that is larger than the true mean element strength. Since design allowables, which are larger than the true mean, lead to unsafe design, this probability is referred as the probability of unsafe design (PUD) herein. PUD is calculated with the N random sets of test results as follows:

$$\mathrm{PUD} = \frac{1}{N}\sum_{i=1}^{N} I(\tau^i_{0.05} > \mu_{e,true}) \quad (18)$$

When we design the truss with the 5th percentile, we expect that the design will have PUD of 5%. However, it is not guaranteed since prior distribution affects the 5th percentile, and the prior distribution is based on an element strength estimate using a failure theory. For example, a prior based on an un-conservative failure theory gives more weight for un-conservative errors than conservative errors. However, the tendency can be reduced by updating the prior with element tests.

4. Assessing the merits of the numbers of coupon and element tests

The objective of this section is to assess the effect of coupon and element tests on reducing uncertainty, estimating design allowables and the corresponding weight penalty. For that purpose, a single set of test results is generated to calculate the 5th percentile and to compute the corresponding weight penalty due to conservativeness. These results are compared with the weight obtained with an infinite number of coupon and element tests. Since the results with a single set of tests are likely to be biased due to sampling error, the above process is repeated 100,000 times to estimate the average weight penalty.

With an infinite number of tests, prediction should be the same with the true element mean, $\mu_{e,true}$, regardless of variability. If a truss member is designed with the 5th percentile for sustaining an axial load F, the weight penalty due to the conservativeness in the 5th percentile is calculated as

$$w_i = \left(\frac{A^i_{0.05}}{A_\infty} - 1\right) \times 100 = \left(\frac{F/\tau^i_{0.05}}{F/\mu_{e,true}} - 1\right) \times 100$$
$$= \left(\frac{\mu_{e,true}}{\tau^i_{0.05}} - 1\right) \times 100\ (\%) \quad (19)$$

where index i represents the ith set of tests results.

When w_i is 3% for (10/5), it means that a design with 10 coupon tests and 5 element tests is 3% heavier than a design with an infinite number of tests. Negative weight penalty indicates that the design is unsafe that the 5th percentile is larger than the true mean.

Fig. 8 illustrates the weight penalty distribution, mean weight penalty, 95% weight penalty and PUD. The 0% weight penalty (the black filled circle) represents design weight with an infinite number of tests.

The mean of the weight penalty (the hollow circle) represents expected conservativeness in the design. The 95% weight penalty (the gray filled circle) represents weight penalty for very conservative designs due to variability in tests. That can be interpreted as that the probability of having more conservative design weight than the 95% weight penalty is 5%. Those measures are calculated from N sets of test results (N=100,000 here) as follows:

$$\mu_{0.05} = \frac{1}{N}\sum_{i=1}^{N} w_i \quad (20)$$

$$P_{0.95} = \frac{1}{N}\sum_{i=1}^{N} I(w_i < w_{0.95}) \quad (21)$$

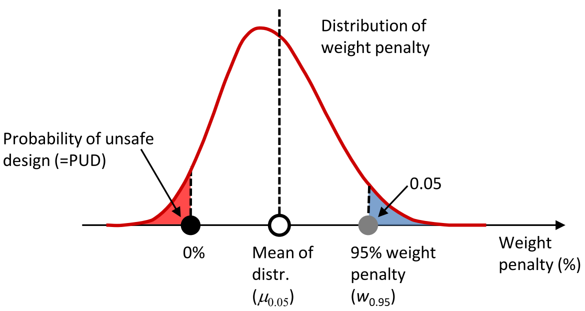

Fig. 8. Distribution of weight penalty due to the variability in tests.

where i is the index of N test sets and the subscript 0.95 in $P_{0.95}$ represents that conservativeness in predicting element strength is 95%.

With N repetitions, N 5th percentiles are collected, and they are varied due to variability in tests. Eq. (20) is to calculate the mean of weight penalty. Eq. (21) is to calculate the 95% percentile of weight penalty of the N sets of test results.

Since a design with a 5th percentile being larger than the true mean element strength is defined as an unsafe design, PUD in Fig. 8 is exactly the same with PUD in Fig. 7 when N sets of test results for two distributions are same.

This procedure needs to be performed for different realizations of epistemic uncertainty. Here, for illustration, we repeat for four cases, 1% and 5% unconservative errors and 1% and 5% conservative errors. These appear to be sufficient to illustrate the effect of different values of epistemic uncertainty.

5. Illustrative example

In this section, the effect of the number of tests is investigated in two steps. First, the conservative mean of the element strength is predicted using a single set of tests, after which the average prediction is estimated with multiple sets of tests.

5.1. The effect of the number of tests with a single set of tests

In this section, estimation of mean element strength is illustrated with a single set of coupon and element tests. The test results were randomly generated from the true distributions defined in Table 2. The difference between the element mean and the coupon mean represents error in the failure theory, as assumed in Eq. (7). Since $k_{3d,calc}=1.0$ is assumed in this paper and $k_{3d,true}=\mu_{e,true}/\mu_{c,true}$ is 0.95, the failure theory overestimates the element strength; that is, the error in the failure theory is unconservative. Randomly generated test results are given in Table 3. The coupon test column presents sample mean and sample standard deviation that will be used to generate coupon samples, and the element test column orderly presents element test results. For example, for 10 coupons and 3 elements (10/3), the mean and standard deviation of coupons were 0.972 and 0.091, respectively, and the first three data, 0.945, 0.955 and 0.987, are used as for three element test results. For four element test results, the first four data, 0.945, 0.955, 0.987 and 0.953 are used. The true distribution is only used for the purpose of simulating tests.

To estimate the mean of element strength, the prior is constructed based on the coupon test results and error bounds as shown in Eqs. (10) and (11). Table 4 gives the assumed error bounds b_e for the mean and $[\sigma_e^{lower}, \sigma_e^{upper}]$ for standard deviation.

Table 2
True distributions of coupon and element tests.

Test	Distribution	Parameters
Coupon test	Normal	$\mu_{c,true}=1.0$, COV 7%
Element test	Normal	$\mu_{e,true}=0.95$, COV 3%

Table 4
Error distributions of element tests.

Error	Distribution	Bounds
b_e	Uniform	$\pm 10\%$
$[\sigma_e^{lower}, \sigma_e^{upper}]$	Uniform	[0, 0.4]

Table 3
Statistics for coupon and element tests.

No. of coupon tests	Coupon test	Element test (order by sequence)
10	$\mu_{c,test}=0.972$, $\sigma_{c,test}=0.091$	0.945, 0.955, 0.987, 0.953, 0.935 $\mu_{e,test}=0.955$, $\sigma_{e,test}=0.0193$
50	$\mu_{c,test}=1.004$, $\sigma_{c,test}=0.073$	0.896, 0.981, 0.939, 0.998, 0.957 $\mu_{e,test}=0.954$, $\sigma_{e,test}=0.039$
90	$\mu_{c,test}=1.001$, $\sigma_{c,test}=0.070$	0.917, 0.989, 0.954, 0.939, 0.948 $\mu_{e,test}=0.949$, $\sigma_{e,test}=0.026$

Table 5
Estimates of the conservative element strength and the corresponding weight penalty (compared to infinite number of tests) from a single set of test results ($\mu_{e,true}=0.95$: Unconservative 5% error in failure theory).

5th percentile					Weight penalty				
n_c	n_e				n_e				
	0	1	3	5		0	1	3	5
10	0.872	0.910	0.936	0.936	10	8.8%	4.3%	1.4%	1.4%
50	0.911	0.893	0.912	0.929	50	4.1%	6.2%	4.0%	2.2%
90	0.910	0.903	0.923	0.927	90	4.3%	5.1%	2.8%	2.3%

Table 6
Four scenarios associated with epistemic uncertainty in failure theory and corresponding example settings (COV of 7% in material strength is assumed).

Magnitude of error in failure theory	Failure theory	True mean of element test	Error bound
Large epistemic uncertainty in failure theory	Unconservative Conservative	$\mu_{e,true}=0.95$ $\mu_{e,true}=1.05$	$b_e=\pm 10\%$ (standard deviation of 5.8%)
Small epistemic uncertainty in failure theory	Unconservative Conservative	$\mu_{e,true}=0.99$ $\mu_{e,true}=1.01$	$b_e=\pm 2\%$ (standard deviation of 1.2%)

Recall that the error bounds represent the current estimate of the maximum error in the failure theory. Detailed procedure of numerical calculation is given in Appendix B.

Table 5 summarizes the 5th percentile value ($\tau_{0.05}$) and the weight penalty after Bayesian update. It is observed that the effect of element tests is more significant than that of coupon tests. As the number of element tests increases between $n_e=1$ and 5, weight penalty decreases from 4–6% to 1.4–2.3%, and a 5th percentile strength converges to 0.95 monotonically. However the effect of the number of coupon tests is ambiguous and no clear trend is observed. This is because the error in the failure theory (Table 4) is much larger than the sampling error in coupons. For the cases of 50 and 90 coupons, $n_e=1$ estimates more conservativeness than $n_e=0$ because the particular element test results happen to be very conservative, as shown in Table 3 (0.896 and 0.917 from a normal distribution with the mean of 0.95 and the standard deviation of 0.0285).

5.2. The effect of the number of tests averaged over multiple sets of tests

The results from the previous subsection depend on the particular samples of coupons and elements. In order to measure the expected effect of tests, the same process is repeated 100,000 times with randomly generated samples, from which 100,000 weight penalties are generated. The effect of the number of tests on the weight penalty is analyzed with three measurements, the mean and 95th percentile of the weight penalty and the probability of unsafe design (PUD). Two scenarios associated with epistemic uncertainty in the failure theory are considered.

The first scenario addresses the effect of relatively large epistemic uncertainty in the failure theory ($b_e=\pm 10\%$) compared to that in coupon samples. With 7% COV in material strength, the uncertainty in the mean coupon strength is small, even with ten coupons. The second scenario examines the effect of relatively small epistemic uncertainty in the failure theory ($b_e=\pm 2\%$). Each scenario is further divided into two cases: unconservative and conservative failure theory. The true mean of the element tests and its error bounds are set to reflect each scenario as shown in Table 6; the other settings are the same as the previous single set example.

The effect of the number of tests is related to the level of uncertainty in the coupon test and the failure theory. Increasing the number of coupon tests can reduce the uncertainty in the coupon test, and the uncertainty in the failure theory can be reduced by increasing the number of element test. Since the uncertainty in predicting the 5th percentile is the combined uncertainty of these uncertainties, the contribution of each test is related to the relative degree of uncertainty. For example, if the uncertainty in the failure theory is larger than that of the coupon test, increasing the number of element tests is more efficient to reduce the combined uncertainty than increasing the number of coupon tests.

When the failure theory has relatively large epistemic uncertainty, the distributions of weight penalties as functions of the number of tests are shown in Fig. 9 for both conservative and unconservative failure theories. $n_c=50$ and $n_e=3$ are assumed as the nominal numbers of tests. The effects of the number of element and coupon tests are shown around the nominal numbers. It is shown that n_e is far more influential than n_c for shifting the distribution to a less conservative region and narrowing it.

With no element tests, the distribution is narrow, since it represents only the sampling uncertainty in 50 coupon tests. As the number of element tests increases, the distribution is first widened for a single element test, because a single test is quite variable, and then gradually narrowed. The updated distribution is also shifted closer to 0% weight penalty. For the unconservative case, Fig. 9(a), the shift is small because the conservativeness in the design with the unconservative failure theory is small. It is unusual that no element test distribution has 0% unsafe design even with un-conservative failure theory. This is because the prior distribution gives very conservative design allowable. As shown in Fig. 5, the prior distribution is similar to the uniform distribution, and the updated distribution is similar to a bell-shaped normal distribution. If the prior and the updated distributions have the same standard deviation, the prior distribution has much conservative 5th percentile than that of the updated distribution, and the design allowable from the prior is much more conservative than that from the updated distribution. For example, 5th percentile of the uniform distribution with standard deviation of 1 is 0.1th percentile of the standard normal distribution. However, for the conservative case, Fig. 9(c), the shift is large since the conservative failure theory provides a very conservative design.

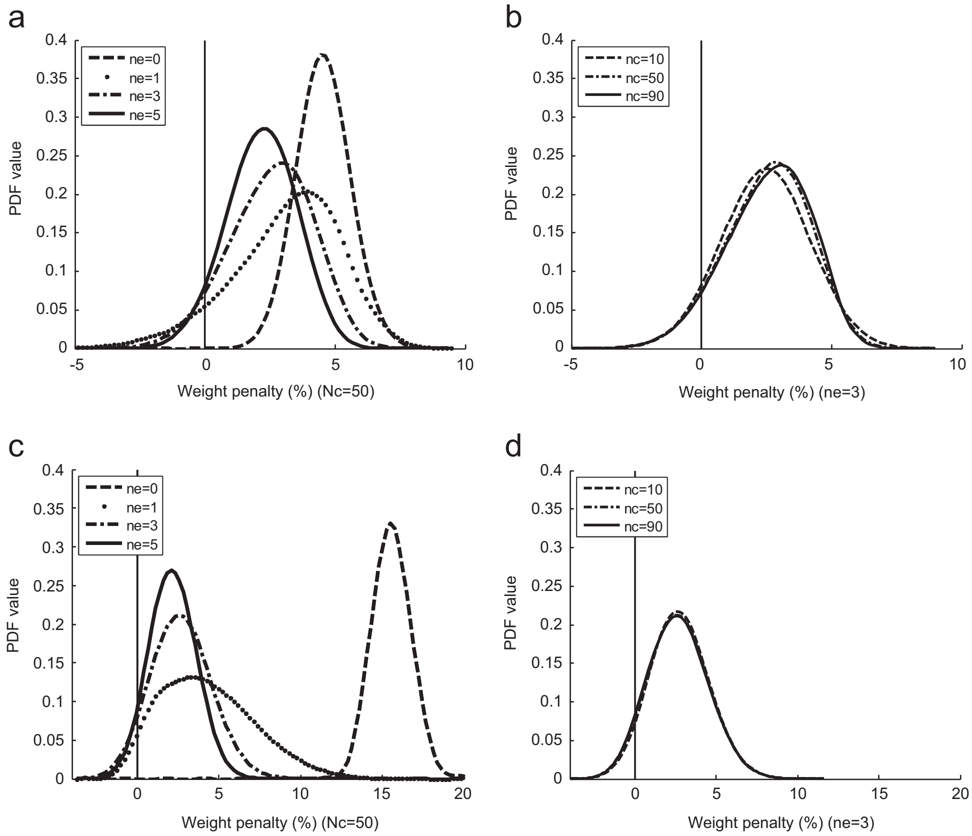

Fig. 9. Distributions of weight penalties for comparison between the number of coupon tests and the number of element tests. (a) $n_e = 0, 1, 3, 5$ with $n_c = 50$ ($\mu_{e,true} = 0.95$ and $b_e = \pm 10\%$), (b) $n_e = 3$ with $n_c = 10,50,90$ ($\mu_{e,true} = 0.95$ and $b_e = \pm 10\%$), (c) $n_e = 0,1,3,5$ with $n_c = 50$ ($\mu_{e,true} = 1.05$ and $b_e = \pm 10\%$) and (d) $n_e = 3$ with $n_c = 10,50,90$ ($\mu_{e,true} = 1.05$ and $b_e = \pm 10\%$).

Table 7
Mean, 95th percentile of weight penalty distribution and probability of unsafe design (PUD) ($\mu_{e,true}$=0.95: unconservative 5% error in failure theory).

	0	1	3	5
	Mean (%)			
10	5.2	3.1	2.5	2.1
50	4.5	3.1	2.5	2.1
90	4.5	3.1	2.5	2.1
	95th Percentile (extreme design weight) (%)			
10	9.3	7.1	5.2	4.3
50	6.2	6.1	5.0	4.3
90	5.7	5.9	5.0	4.3
	Probability of unsafe design (PUD) (%)			
10	2	10	8	7
50	0	9	7	6
90	0	9	7	6

Table 8
Mean, 95th percentile of weight penalty distribution and probability of unsafe design (PUD) ($\mu_{e,true}$=1.05: Conservative 5% error in failure theory).

	0	1	3	5
	Mean (%)			
10	16.3	4.3	2.7	2.2
50	15.5	4.2	2.6	2.1
90	15.4	4.2	2.6	2.1
	95th Percentile (extreme design weight) (%)			
10	20.8	9.3	5.8	4.5
50	17.4	9.5	5.7	4.5
90	16.8	9.5	5.7	4.5
	Probability of unsafe design (PUD) (%)			
10	0	5	6	6
50	0	5	7	6
90	0	5	7	6

Tables 7 and 8 summarize the distributions with three statistics–mean weight penalty, 95th percentile and PUD-in terms of the number of tests.

We first consider the case of minimal testing with only 10 coupon tests and no element tests. For the case of unconservative failure theory, shown in Table 7, minimal testing will cost us a 5.2% weight penalty on average, and a 2% chance that we will end up with unconservative design. For the case of conservative failure theory, in Table 8, the weight penalty shoots up to 16.3% and we do not run the chance of unconservative design. The weight penalties with the 95th percentiles (corresponding to tests that happen to be on the conservative side) are about 10% higher.

With a single element test, the weight penalty drops significantly to 3.1% for the unconservative failure theory, in Table 7, and to 4.3% for the conservative case in Table 8. However, with only a single element test, we have a much higher chance of unconservative design: PUD of 10% and 5%, respectively. This is because the characteristics of failure theory is reflected on the prior and PUD. For un-conservative failure theory with 5% error, in Table 7, PUDs with a single element test are less than 10%, and they converge to 5% as the number of element tests increases. For conservative failure theory with 5% error, PUDs with a single element test are 5%, which are more conservative than PUDs with un-conservative failure theory presented in Table 7 as expected. However, the PUD is increased to 7% at 3 elements and then decreased to 6% for 5 elements. This unexpected behavior is related to the prior distribution with non-Gaussian shape, which is shown in Fig. 5, but PUDs always converge to 5% as the number of element tests increases. For no element tests, PUD is very close to zero. This is because of the difference in the shape of distribution. The 5th percentile of prior, which resembles a uniform distribution, is much more conservative than that of posterior distribution updated once with a single element test, which resembles Gaussian distribution. For example, the 5th percentile of a uniform distribution with standard deviation of 1 is 0.1th percentile of a standard normal distribution. The weight penalties continue to drop substantially and the PUDs converge to 5% with more element tests. On the other hand, the effect of adding coupon tests is much smaller, and increasing coupon tests from 50 to 90 hardly make any difference.

The fact that, for this example, element tests are more important than coupon tests can be understood by observing the magnitude of two epistemic uncertainties. The variability in the coupon strength is 7% (see Table 2), so even with 10 coupon tests, the standard deviation of the mean coupon strength is only 2.2%, which is epistemic uncertainty in sampling. On the other hand, with ±10% error bounds, the standard deviation of the epistemic uncertainty in the failure theory is 5.8%. This is why element tests were more significant in reducing uncertainty. If, on the other hand, the failure theory was much more accurate, then element tests are expected to be less significant. For example, with ±2% error bounds, the magnitude of the epistemic uncertainty in failure theory is merely 1.2%. With such an accurate failure theory, it turns out that the number of coupon tests becomes more influential than the number of element tests.

It turned out that increasing the number of element tests is more important than increasing the number of coupon tests when we have the large epistemic uncertainty (±10%) in the failure theory. However, when the epistemic uncertainty is small (±2%), the number of coupon tests becomes more influential than the number of element tests. In parallel to Fig. 9, Fig. 10 shows a comparison between the effect of n_c and n_e on the weight penalty when the error in the failure theory is small. It is clearly seen that the effect of the number of coupon tests is more influential than the number of element tests for decreasing the chance of having very conservative designs and reducing the variation of design.

Compared to Tables 7 and 8, the increased accuracy of the failure theory reduces substantially the penalty associated with no

Table 9
Mean, 95th percentile of weight penalty distribution and probability of unsafe design (PUD) ($\mu_{e,true}$=0.99: unconservative 1% error in failure theory).

	0	1	3	5
	Mean (%)			
10	3.2	3.0	2.4	2.0
50	1.5	1.5	1.4	1.3
90	1.3	1.2	1.2	1.1
	95th Percentile (extreme design weight) (%)			
10	7.4	6.8	5.4	4.3
50	3.2	3.1	2.9	2.7
90	2.5	2.5	2.4	2.3
	Probability of unsafe design (PUD) (%)			
10	10	9	8	7
50	7	6	7	7
90	5	5	6	6

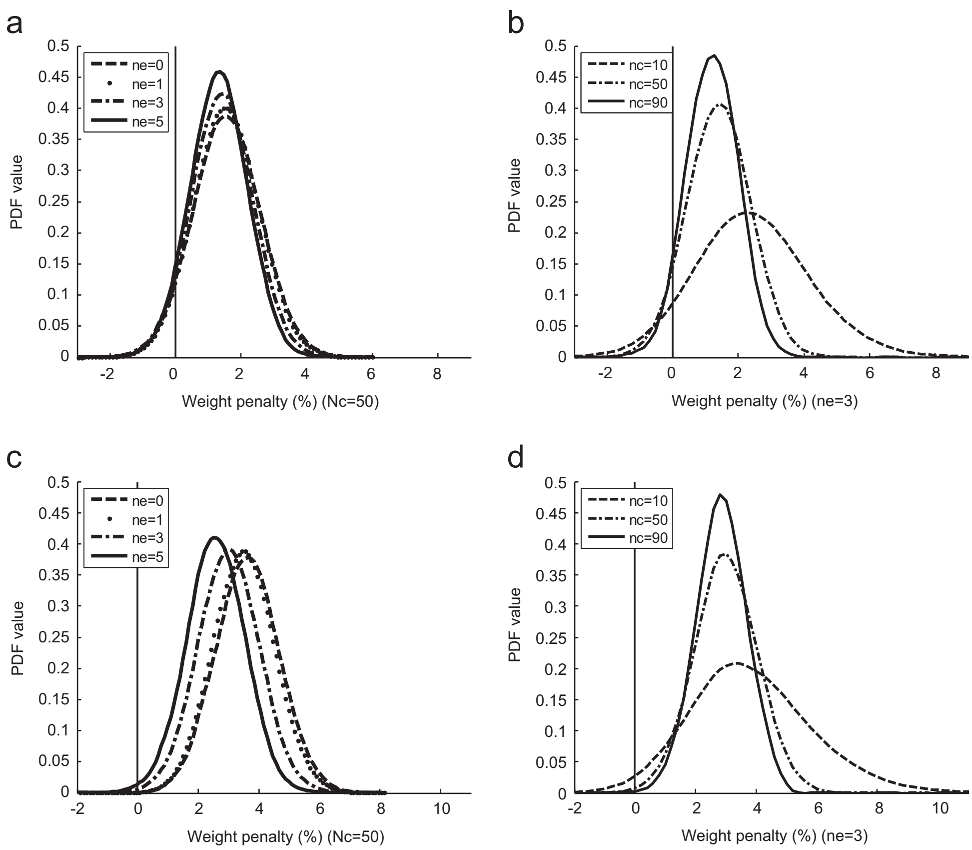

Fig. 10. Distributions of weight penalties for comparison between the number of coupon tests and the number of element tests. (a) n_e = 0,1,3,5 with n_c = 50 ($\mu_{e,true}$ = 0.99 and b_e = ±2%), (b) n_e = 3 with n_c = 10,50,90 ($\mu_{e,true}$ = 0.99 and b_e = ±2%), (c) n_e = 0,1,3,5 with n_c = 50 ($\mu_{e,true}$ = 1.01 and b_e = ±2%) and (d) n_e = 3 with n_c = 10,50,90 ($\mu_{e,true}$ = 1.01 and b_e = ±2%).

Table 10
Mean, 95th percentile of weight penalty distribution and probability of unsafe design (PUD) ($\mu_{e,true}=1.01$: conservative 1% error in failure theory).

	0	1	3	5
	Mean (%)			
10	5.3	4.9	3.8	2.9
50	3.6	3.4	3.0	2.5
90	3.3	3.2	2.8	2.5
	95th Percentile (extreme design weight) (%)			
10	9.6	8.9	7.3	5.8
50	5.3	5.1	4.6	4.1
90	4.6	4.4	4.1	3.7
	Probability of unsafe design (PUD) (%)			
10	2	2	2	3
50	1	1	1	2
90	0	0	1	1

Table 11
Probability of Z at two different values (means and COVs were obtained with 1000 repetitions of probability calculation with 1 million samples).

Z value		0.955	0.975
MCS	Mean	2.34×10^{-7}	6.77×10^{-4}
	COV (%)	210.7	3.9
Convolution integral		2.40×10^{-7}	6.78×10^{-4}

element test. For 10 coupon tests, the weight penalty for no element test is reduced from 5.2% to 3.2% for unconservative error (Tables 7 and 9) and from 16.3% to 5.3% for conservative error (Tables 8 and 10). Also, because the epistemic uncertainties associated with the failure theory are not comparable to that in the mean of coupon tests, the contributions of the number of coupon and element tests become comparable. Increasing element tests from 1 to 5 for 10 coupon tests reduces the weight penalty from 3% to 2% (Table 9) and from 4.9% to 2.9% (Table 10). In comparison, for one element test, increasing the number of coupon tests from 10 to 90 reduces the weight penalty from 3% and 4.9% to 1.2% and 3.3%, respectively.

For an un-conservative failure theory with 1% error, shown in Table 9, PUDs increasingly converges to 5% from 1 to 5 element tests. For a conservative failure theory with 1% error, Table 10, PUDs increase as the number of element tests increases, and the conservative prediction characteristics of failure theory is weaken as the number of element tests increases, but the characteristics still remains with 5 element tests.

The above examples illustrate the effect of the number of tests to predict a design allowable using Bayesian inference. In the current design practice, the lowest element test result is used as a design allowable. A comparison between the two approaches is shown in Appendix C.

5.3. Accuracy of a convolution integral on calculating a conditional distribution

It has been shown that double-loop MCS can be used to calculate the distribution in Eqs. (5) and (10) as shown in Figs. 2 and 4. However, MCS has a computational challenge in the tail region (low-probability region) as well as sampling error. For example, a 10^{-4} level of probability can be hardly estimated with 10,000 samples. Different from MCS, a convolution integral can calculate a nearly exact distribution without having sampling errors. In this section, the accuracy of the convolution integral is compared with that of MCS.

In order to illustrate the advantage of the convolution integral, the probability of the product of two random variables, $\hat{Z}=\hat{X}\times\hat{Y}$, are used. It is assumed that the two independent random variables are defined as $\hat{X}\sim N(1.1, 0.0096)$ and $\hat{Y}\sim U(0.9, 1.1)$. For MCS, one million samples are used to evaluate the probability of Z values at 0.955 and 0.975. Since MCS has sampling error, this process is repeated 1000 times; the mean and standard deviation are listed in Table 11. For the convolution integral, the entire range is divided by 50 segments, and three-point Gauss quadrature in each segment is used in integrating Eq. (9) with $b_e=0.1$, $\mu_{c,test}=1.1$, and $\sigma_{c,test}/\sqrt{n_c}=0.0096$. The results only differ by 0.2% when 400 segments are used. Different from MCS, there is no need for repetition because convolution integration does not have sampling error.

When the probability is of the order of 10^{-4}, MCS has about a 3.9% coefficient of variation (COV), while the convolution integral shows a very little calculation error. When the probability is to the order of 10^{-7}, the MCS with 1 million samples is not meaningful, as reflected in the COV value of 210%. However, the convolution integral is still accurate, and the value can be obtained by a one-time calculation. Note that the estimated error in the mean PF with 1000 repetitions can be calculated as $4.93\times10^{-7}/1000^{0.5}=1.56\times10^{-8}$.

6. Conclusions

In this paper, the effect of the number of coupon and element tests on reducing conservativeness and weight penalties due to the uncertainty in structural element strength was studied. Two sources of epistemic uncertainties were considered: (a) the sampling uncertainty in measuring material variability and (b) the uncertainty in the failure theory. A large number of coupons reduce the uncertainty in measuring material variability, while element tests reduce the uncertainty in the failure theory. These uncertainties were combined using the convolution integral, which is more accurate and robust than MCS. Then, Bayesian inference was used to update this uncertainty with element test results. Because test results can vary, a large number of simulations were used to obtain mean performance and distributions.

For a typical case of $\pm10\%$ uncertainty bounds on the failure theory, 5% actual error, 7% and 3% coefficient of variation in material strength and element strength, element tests were found to be very important in reducing weight penalties from about 15% with no tests, to about 2% with five element tests. The effect of the number of coupon tests was much smaller because sampling uncertainty was much smaller than the uncertainty in the failure theory. When the failure theory was much more accurate ($\pm2\%$ confidence bounds and 1% actual error), the effect of the number of coupons became comparable to that of element tests. The methodology developed would thus allow designers to estimate the weight benefits of tests and improvements in failure predictions.

Acknowledgment

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Appendix A. Statistical formulation of possible true distribution of mean and standard deviation of element strength

The PTD of element mean failure strength can be expressed as

$$f_{\mu e,Ptrue}(\mu_{e,Ptrue})=\int_{-\infty}^{\infty} f_{\mu e,Ptrue}(\mu_{e,Ptrue}|\mu_{c,Ptrue})f_{\mu c,Ptrue}(\mu_{c,Ptrue})\,d\mu_{c,Ptrue} \tag{A1}$$

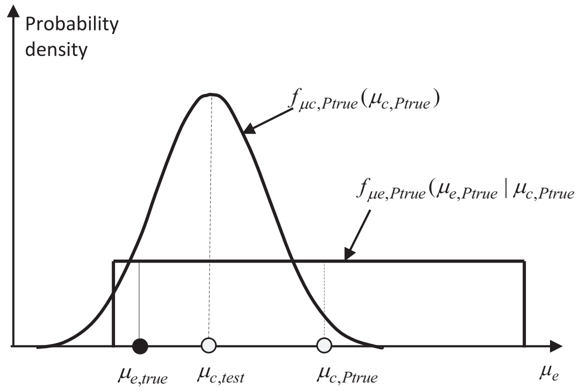

Fig. A1. The possible true distribution of mean failures strength of specimens and the conditional distribution of the element mean failure strength.

which is in the form of the convolution integral. The conditional PDF $f_{\mu e,Ptrue}(\mu_{e,Ptrue}|\mu_{c,Ptrue})$ corresponds to the distribution of $\hat{k}_{3d,Ptrue}$. In the following, the two PDFs in the integrand will be explained.

In this paper, $k_{3d,calc}=1$ is used for simplicity, and it is assumed that $e_{k,Ptrue}$ follows a uniform distribution with bounds $\pm b_e$ as

$$f_{k,Ptrue}(e_{k,Ptrue}) = \begin{cases} \frac{1}{2b_e} & \text{if } |e_{k,Ptrue}| \le b_e \\ 0 & \text{otherwise} \end{cases} \quad \text{(A2)}$$

By using Eq. (A2), $f_{\mu e,Ptrue}(\mu_{e,Ptrue})$ can be obtained from all possible combinations of random variables generated from $f_{k,Ptrue}(e_{k,Ptrue})$ and $f_{\mu c,Ptrue}(\mu_{c,Ptrue})$. For a given sample of $\mu_{c,Ptrue}$, the PTD of element failure strength can be regarded as a conditional PDF $f_{\mu_{e,Ptrue}}(\mu_{e,Ptrue}|\mu_{c,Ptrue})$, which is a uniform distribution with a width of $2b_e$ and mean at $\mu_{c,Ptrue}$.

$$f_{\mu e,Ptrue}(\mu_{e,Ptrue}|\mu_{c,Ptrue}) = \begin{cases} \frac{1}{2b_e\mu_{c,Ptrue}} & \text{if } \left|\frac{\mu_{e,Ptrue}}{\mu_{c,Ptrue}}-1\right| \le b_e \\ 0 & \text{otherwise} \end{cases} \quad \text{(A3)}$$

The PDF in Eq. (A3) represents the epistemic uncertainty in failure theory. The PTD $f_{\mu e,Ptrue}(\mu_{e,Ptrue})$ can be calculated by considering all possible values of $\mu_{c,Ptrue}$ with Eq. (A3).

By using Eq. (A3), PDF of the PTD of $\mu_{c,Ptrue}$ is calculated from coupon test results as

$$f_{\mu c,Ptrue}(\mu_{c,Ptrue}) = \varphi\left(\mu_{c,Ptrue}|\mu_{c,test}, \frac{\sigma_{c,test}}{\sqrt{n_c}}\right) \quad \text{(A4)}$$

where the notation $\varphi(x|a,b)$ denotes the value of normal PDF with mean a and standard deviation b at x. Samples of $\mu_{c,Ptrue}$ are generated from Eq. (A4), which are then used in Eq. (A3) to generate samples of $\mu_{e,Ptrue}$. Fig A1 illustrates the conditional PDF of $\mu_{e,Ptrue}$ for a given sample of $\mu_{c,Ptrue}$, which is drawn from $f_{\mu c,Ptrue}(\mu_{c,Ptrue})$ based on $\mu_{c,test}$. Note that $\mu_{e,true}$ is given as a unique value and is covered by the PTD $f_{\mu e,Ptrue}(\mu_{e,Ptrue}|\mu_{c,Ptrue})$.

With Eqs. (A3) and (A4), the convolution integral in Eq. (A2) can be directly integrated as

$$f_{\mu e,Ptrue}(\mu_{e,Ptrue}) = \int_{\frac{\mu_{e,Ptrue}}{(1-b_e)}}^{\frac{\mu_{e,Ptrue}}{(1+b_e)}} \frac{1}{2b_e\mu_{c,Ptrue}} \phi\left(\mu_{c,Ptrue}|\mu_{c,test}, \frac{\sigma_{c,test}}{\sqrt{n_c}}\right) d\mu_{c,Ptrue} \quad \text{(A5)}$$

The PDF in Eq. (A5) is a prior distribution of mean failure strength of elements, which includes the effect of uncertainty from failure theory as well as that of a finite number of samples.

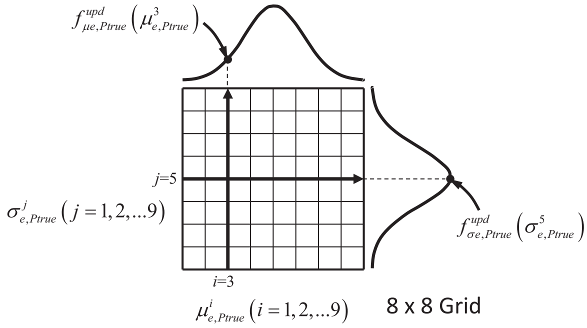

Fig. B1. A 8×8 grid for obtaining a joint PDF and its marginal PDFs.

Appendix B. Numerical scheme to obtain the presented results

For the mean element strength, a range of [0.78, 1.22] was found to be large enough to capture the updated joint probability distribution, because the initial distribution for the mean element strength has very little influence on posterior distribution on both tails. Fig. 5 shows a typical shape of the initial distribution for the element mean. The standard deviation is bounded in [0, 0.4], as noted in Table 4. In order to calculate the updated distribution from Bayesian inference, each range is discretized into 200 equal intervals, and this discretization generates a 200×200 grid. The updated joint PDF is calculated at each grid point using Eq. (12). Then, the prior is updated using a likelihood function with different numbers of element tests; i.e. $n_e=1$, 3, and 5.

The marginal updated distributions are obtained using the updated joint distribution as expressed in Eqs. (15) and (16). For the updated marginal element mean distribution, conditional PDFs for a given 201 mean element strength are integrated over 201 points using Gaussian quadrature with 2 points. Fig. B1 shows an equivalent example that has an 8×8 grid. The abscissa and ordinate of the grid are mean and standard deviation, respectively. The superscripts i and j are the horizontal and vertical coordinates of the grid. For example, $\mu^3_{e,Ptrue}$ is the value of the mean on the third vertical line. The marginal distribution of the updated mean element strength is formed by calculating PDF values on 9 given mean values. $f^{upd}_{\mu e,Ptrue}(\mu^3_{e,Ptrue})$ is equal to a value obtained by integrating a conditional PDF of the standard deviation for $\mu_{e,Ptrue} = \mu^3_{e,Ptrue}$ over the vertical arrow.

Appendix C. A comparison between current statistical method and the proposed Bayesian method

In the building-block process, the number of element tests is usually limited to three, due to the large number of structural elements to test [7,8]. In the current practice, analytical prediction of element strength based on failure theory is modified by applying the lowest ratio between the test results and the predicted failure stress. This process can be interpreted as applying an implicit knockdown factor to the average test failure stress to obtain a design allowable of element strength [7]. If the tests are repeated with different elements, the predicted failure stress will be changed as well as the implicit knockdown factor.

In this paper, we propose a way to estimate a design allowable by adding certain conservativeness on estimated mean element strength using Bayesian inference. Bayesian inference has a strong point to combine information from different sources [17]. In the proposed method, we combine confidence interval information from analytical prediction (prior) with data from element tests while the current practice relies on data from element tests.

Table C1
Probability of unsafe design (PUD), 95th percentile of weight penalties and magnifier.

Case	Proposed method			Current method
	$N_c=10$	$N_c=50$	$N_c=90$	
	Probability of unsafe design (PUD) (%)			
Unconservative 5% error	6.1	6.4	6.4	12.5
Conservative 5% error	6.7	7.3	7.3	12.5
Unconservative 1% error	5.8	0.4	0.0	12.5
Conservative 1% error	7.8	0.6	0.2	12.5
	95th Percentile of weight penalty factor (extreme design weight) (%)			
Unconservative 5% error	5.4	5.2	5.1	6.8
Conservative 5% error	5.7	5.7	5.7	6.8
Unconservative 1% error	5.6	4.2	3.9	6.8
Conservative 1% error	6.1	4.3	3.9	6.8

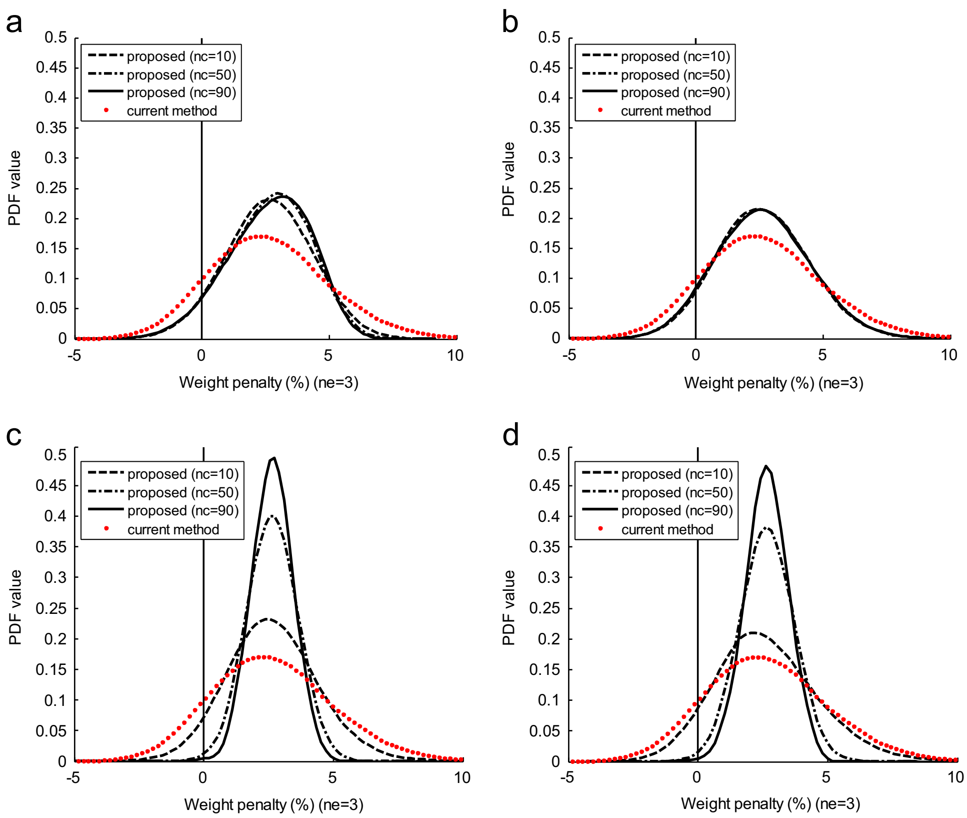

Fig. C1. Distributions of weight penalties for comparison between the proposed method and current method (taking the lowest element strength among three). (a) Unconservative 5% error ($\mu_{e,true}=0.95$ and $b_e=\pm10\%$), (b) conservative 5% error ($\mu_{e,true}=1.05$ and $b_e=\pm10\%$), (c) unconservative 5% error ($\mu_{e,true}=0.99$ and $b_e=\pm2\%$), (d) conservative 5% error ($\mu_{e,true}=1.01$ and $b_e=\pm2\%$),

To compare the two approaches, the mean weight penalties are matched, by which we can compare the achieved safety for the same weight penalty in terms of PUD. For the conventional method, the lowest element strength out of three is taken as the conservative element strength so that the probability of being larger than the true mean is 12.5%.

$$\Pr(\tau_{lowest}\geq\mu_{e,true})=0.125 \quad (C1)$$

In the proposed method, since the conservative element strength is the 5th percentile of the mean element strength, $\tau_{0.05}$, the probability can be calculated as

$$P_{Bayes}=\Pr(k\tau_{0.05}\leq\mu_{e,true}) \quad (C2)$$

Note that the constant k is used to match their mean weight penalties. As we do in the previous examples, the weight penalties for both methods are calculated using MCS with 100,000 samples.

$$\text{For current method}\quad w_{i,curr}=\left(\frac{\mu_{e,true}}{\tau_{lowest}}-1\right)\times 100\ (\%) \quad (C3)$$

$$\text{For the proposed method}\ w_i=\left(\frac{\mu_{e,true}}{k\tau^{i}_{0.05}}-1\right)\times 100\ (\%) \quad (C4)$$

where i represents the index of samples, and the subscript 'curr' represents the current method.

Table C1 presents the mean and 95th percentile of weight penalty for the proposed and current method. Both methods use three element tests, $N_c=3$. Since the proposed method combines the information from analytical prediction with data from coupon, the effect of the number of coupon tests is shown.

Fig. C1 shows the distributions of weight penalty for the proposed and current method, and Table C1 characterizes the distribution with PUD representing safety and 95 percentile of weight penalty representing extreme cases for both methods. The proposed method shows better results in both measures. For PUD, the proposed method has at least 5% less PUD than the current method; that is, the proposed method is safer than the current method by at least by 5%. For 95th percentile of weight penalty, the proposed method is less at least 0.7% and at most 3%. For cases of 5% errors in prediction, the number of coupon test is very limited.

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