Introduction

• Linear systems
  - Infinitesimal deformation: no significant difference between the deformed and undeformed shapes
  - Stress and strain are defined in the undeformed shape
  - The weak form is integrated over the undeformed shape
• Large deformation problem
  - The difference between the deformed and undeformed shapes is large enough that they cannot be treated the same
  - The definitions of stress and strain should be modified from the assumption of small deformation
  - The relation between stress and strain becomes nonlinear as deformation increases
• This chapter will focus on how to calculate the residual and tangent stiffness for a nonlinear elasticity model
Introduction

- **Frame of Reference**
  - The weak form must be expressed based on a frame of reference
  - Often initial (undeformed) geometry or current (deformed) geometry are used for the frame of reference
  - proper definitions of stress and strain must be used according to the frame of reference
- **Total Lagrangian Formulation**: initial (undeformed) geometry as a reference
- **Updated Lagrangian Formulation**: current (deformed) geometry
- Two formulations are theoretically identical to express the structural equilibrium, but numerically different because different stress and strain definitions are used

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3.2 Stress and Strain Measures

Goals – Stress & Strain Measures

- Definition of a nonlinear elastic problem
- Understand the deformation gradient?
- What are Lagrangian and Eulerian strains?
- What is polar decomposition and how to do it?
- How to express the deformation of an area and volume
- What are Piola-Kirchhoff and Cauchy stresses?
Mild vs. Rough Nonlinearity

• **Mild** Nonlinear Problems (Chap 3)
  - Continuous, *history-independent* nonlinear relations between stress and strain
  - Nonlinear elasticity, Geometric nonlinearity, and deformation-dependent loads

• **Rough** Nonlinear Problems (Chap 4 & 5)
  - Equality and/or inequality constraints in constitutive relations
  - *History-dependent* nonlinear relations between stress and strain
  - Elastoplasticity and contact problems

What Is a Nonlinear Elastic Problem?

• **Elastic** (same for linear and nonlinear problems)
  - Stress-strain relation is elastic
  - Deformation disappears when the applied load is removed
  - Deformation is history-independent
  - Potential energy exists (function of deformation)

• **Nonlinear**
  - Stress-strain relation is nonlinear (\(D\) is not constant or do not exist)
  - Deformation is large

• **Examples**
  - Rubber material
  - Bending of a long slender member (small strain, large displacement)
Reference Frame of Stress and Strain

- Force and displacement (vector) are independent of the configuration frame in which they are defined (Reference Frame Indifference)
- Stress and strain (tensor) depend on the configuration
- **Total Lagrangian or Material Stress/Strain**: when the reference frame is undeformed configuration
- **Updated Lagrangian or Spatial Stress/Strain**: when the reference frame is deformed configuration
- Question: What is the reference frame in linear problems?

Deformation and Mapping

- Initial domain $\Omega_0$ is deformed to $\Omega_x$
  - We can think of this as a mapping from $\Omega_0$ to $\Omega_x$
- $X$: material point in $\Omega_0$
- $x$: material point in $\Omega_x$
- Material point $P$ in $\Omega_0$ is deformed to $Q$ in $\Omega_x$

\[
\mathbf{x} = \mathbf{X} + \mathbf{u} \quad \Longleftrightarrow \quad \mathbf{x} = \Phi(\mathbf{X}, t) = \mathbf{X} + \mathbf{u}(\mathbf{X}, t)
\]

$\Phi, \Phi^{-1}$: One-to-one mapping
Continuously differentiable
Deformation Gradient

- Infinitesimal length \( dX \) in \( \Omega_0 \) deforms to \( dx \) in \( \Omega_x \)
- Remember that the mapping is continuously differentiable

\[
dx = \frac{\partial x}{\partial X} dX \quad \Rightarrow \quad dx = F dX
\]

- Deformation gradient:

\[
F_{ij} = \frac{\partial x_i}{\partial X_j}
\]
\[
F = 1 + \frac{\partial u}{\partial X} = 1 + \nabla_0 u
\]
\[
1 = [\delta_{ij}],
\]
\[
\nabla_0 = \frac{\partial}{\partial X}, \quad \nabla_x = \frac{\partial}{\partial x}
\]

- Gradient of mapping \( \Phi \)
- Second-order tensor, Depend on both \( \Omega_0 \) and \( \Omega_x \)
- Due to one-to-one mapping: \( \det F \equiv J > 0. \) \( dx = F^{-1} dX \)
- \( F \) includes both deformation and rigid-body rotation

Example – Uniform Extension

- Uniform extension of a cube in all three directions

\[
x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3
\]

- Continuity requirement: \( \lambda_i > 0 \) Why?
- Deformation gradient:

\[
F = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix}
\]

- \( \lambda_1 = \lambda_2 = \lambda_3 \): uniform expansion (dilatation) or contraction
- Volume change
  - Initial volume: \( dV_0 = dX_1 dX_2 dX_3 \)
  - Deformed volume:

\[
dV_x = dx_1 dx_2 dx_3 = \lambda_1 \lambda_2 \lambda_3 dX_1 dX_2 dX_3 = \lambda_1 \lambda_2 \lambda_3 dV_0
\]
Green-Lagrange Strain

- Why different strains?
- Length change: \( \|dx\|^2 - \|dX\|^2 = dx^T dx - dX^T dX \)
  \[ = dX^T F^T F dX - dX^T dX \]
  \[ = dX^T (F^T F - I) dX \]
  Ratio of length change

- Right Cauchy-Green Deformation Tensor
  \[ C = F^T F \]

- Green-Lagrange Strain Tensor
  \[ E = \frac{1}{2} (C - I) \]
  The effect of rotation is eliminated
  To match with infinitesimal strain

Green-Lagrange Strain cont.

- Properties:
  - \( E \) is symmetric: \( E^T = E \)
  - No deformation: \( F = I, E = 0 \)
  \[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) \]
  \[ E = \frac{1}{2} \left( \frac{\partial u}{\partial X} + \frac{u^T}{\partial X} + \frac{u^T}{\partial X} \right) \]
  \[ = \frac{1}{2} \left( \nabla_0 u + \nabla_0 u^T + \nabla_0 u^T \nabla_0 u \right) \]
  Displacement gradient
  Higher-order term

- When \( |\nabla_0 u| \ll 1 \), \( E \approx \frac{1}{2} (\nabla_0 u + \nabla_0 u^T) = \varepsilon \)

- \( E = 0 \) for a rigid-body motion, but \( \varepsilon \neq 0 \)
Example – Rigid-Body Rotation

- Rigid-body rotation
  \[ x_1 = X_1 \cos \alpha - X_2 \sin \alpha \]
  \[ x_2 = X_1 \sin \alpha + X_2 \cos \alpha \]
  \[ x_3 = X_3 \]

- Approach 1: using deformation gradient
  \[ \mathbf{F} = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
  \[ \mathbf{F}^T \mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
  \[ E = \frac{1}{2} (\mathbf{F}^T \mathbf{F} - 1) = 0 \]

  Green-Lagrange strain removes rigid-body rotation from deformation

Example – Rigid-Body Rotation cont.

- Approach 2: using displacement gradient
  \[ u_1 = x_1 - X_1 = X_1 \cos \alpha - 1 - X_2 \sin \alpha \]
  \[ u_2 = x_2 - X_2 = X_1 \sin \alpha + X_2 \cos \alpha - 1 \]
  \[ u_3 = x_3 - X_3 = 0 \]
  \[ \nabla_0 \mathbf{u} = \begin{bmatrix} \cos \alpha - 1 & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
  \[ \nabla_0 \mathbf{u}^T \nabla_0 \mathbf{u} = \begin{bmatrix} 2(1 - \cos \alpha) & 0 & 0 \\ 0 & 2(1 - \cos \alpha) & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
  \[ E = \frac{1}{2} (\nabla_0 \mathbf{u} + \nabla_0 \mathbf{u}^T + \nabla_0 \mathbf{u}^T \nabla_0 \mathbf{u}) = 0 \]
Example – Rigid-Body Rotation cont.

• What happens to engineering strain?

\[ u_1 = x_1 - X_1 = X_1 (\cos \alpha - 1) - X_2 \sin \alpha \]
\[ u_2 = x_2 - X_2 = X_1 \sin \alpha + X_2 (\cos \alpha - 1) \]
\[ u_3 = x_3 - X_3 = 0 \]

\[
\varepsilon = \begin{bmatrix}
\cos \alpha - 1 & 0 & 0 \\
0 & \cos \alpha - 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Engineering strain is unable to take care of rigid-body rotation

Eulerian (Almansi) Strain Tensor

• Length change: \( \|dx\|^2 - \|dX\|^2 = dx^Tdx - dX^TdX \)
  \[ = dx^Tdx - dx^TF^{-T}F^{-1}dx \]
  \[ = dx^T(1 - F^{-T}F^{-1})dx \]
  \[ = dx^T(1 - b^{-1})dx \]

• Left Cauchy-Green Deformation Tensor

\[ b = FF^T \]

\[ b^{-1}: \text{Finger tensor} \]

• Eulerian (Almansi) Strain Tensor

\[ e = \frac{1}{2}(1 - b^{-1}) \]

Reference is deformed (current) configuration
Eulerian Strain Tensor cont.

- Properties
  - Symmetric
  - Approach engineering strain when $\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \ll 1$
  - In terms of displacement gradient

$$
\mathbf{e} = \frac{1}{2} \left( \frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}^T}{\partial \mathbf{x}} - \frac{\partial \mathbf{u}^T}{\partial \mathbf{x}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)
$$

Spatial gradient

- Relation between $\mathbf{E}$ and $\mathbf{e}$

$$
\mathbf{E} = \mathbf{F}^T \mathbf{e} \mathbf{F}
$$

Example - Lagrangian Strain

- Calculate $\mathbf{F}$ and $\mathbf{E}$ for deformation in the figure

- Mapping relation in $\Omega_0$

$$
\begin{align*}
X &= \sum_{I=1}^{4} N_I(s,t)X_I = \frac{3}{4}(s + 1) \\
Y &= \sum_{I=1}^{4} N_I(s,t)Y_I = \frac{1}{2}(t + 1)
\end{align*}
$$

- Mapping relation in $\Omega_x$

$$
\begin{align*}
x(s,t) &= \sum_{I=1}^{4} N_I(s,t)x_I = 0.35(1 - t) \\
y(s,t) &= \sum_{I=1}^{4} N_I(s,t)y_I = s + 1
\end{align*}
$$
Example – Lagrangian Strain cont.

• Deformation gradient

\[
F = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial \mathbf{s}} \frac{\partial \mathbf{s}}{\partial \mathbf{X}} = \begin{bmatrix} 0 & -0.35 \ 1 & 0 \ 0 & -0.7 \ 4/3 & 0 \end{bmatrix} = \begin{bmatrix} 4/3 & 0 \ 0 & 2 \end{bmatrix}
\]

• Green-Lagrange Strain

\[
E = \frac{1}{2} (F^T F - I) = \begin{bmatrix} 0.389 & 0 \ 0 & -0.255 \end{bmatrix}
\]

Tension in \( X_1 \) dir.
Compression in \( X_2 \) dir.

Example – Lagrangian Strain cont.

• Almansi Strain

\[
b = F \cdot F^T = \begin{bmatrix} 0.49 & 0 \ 0 & 1.78 \end{bmatrix}
\]

\[
e = \frac{1}{2} (I - b^{-1}) = \begin{bmatrix} -0.52 & 0 \ 0 & 0.22 \end{bmatrix}
\]

Compression in \( x_1 \) dir.
Tension in \( x_2 \) dir.

• Engineering Strain

\[
\nabla_0 u = F - I = \begin{bmatrix} -1 & -0.7 \ 1.33 & -1 \end{bmatrix}
\]

\[
\varepsilon = \frac{1}{2} (\nabla_0 u + \nabla_0 u^T) = \begin{bmatrix} -1 & 0.32 \ 0.32 & -1 \end{bmatrix}
\]

Artificial shear deform.
Inconsistent normal deform.

Which strain is consistent with actual deformation?
Example - Uniaxial Tension

- Uniaxial tension of incompressible material ($\lambda_1 = \lambda > 1$)
- From incompressibility
  
  \[
  \lambda_1\lambda_2\lambda_3 = 1 \implies \lambda_2 = \lambda_3 = \lambda^{-1/2}
  \]

- Deformation gradient and deformation tensor
  \[
  F = \begin{bmatrix}
  \lambda & 0 & 0 \\
  0 & \lambda^{-1/2} & 0 \\
  0 & 0 & \lambda^{-1/2}
  \end{bmatrix},
  \qquad C = \begin{bmatrix}
  \lambda^2 & 0 & 0 \\
  0 & \lambda^{-1} & 0 \\
  0 & 0 & \lambda^{-1}
  \end{bmatrix}
  \]

- G-L Strain
  \[
  E = \frac{1}{2} \begin{bmatrix}
  \lambda^2 - 1 & 0 & 0 \\
  0 & \lambda^{-1} - 1 & 0 \\
  0 & 0 & \lambda^{-1} - 1
  \end{bmatrix}
  \]

Example - Uniaxial Tension

- Almansi Strain ($b = C$)
  
  \[
  b^{-1} = \begin{bmatrix}
  \lambda^{-2} & 0 & 0 \\
  0 & \lambda & 0 \\
  0 & 0 & \lambda
  \end{bmatrix},
  \quad e = \frac{1}{2} \begin{bmatrix}
  1 - \lambda^{-2} & 0 & 0 \\
  0 & 1 - \lambda & 0 \\
  0 & 0 & 1 - \lambda
  \end{bmatrix}
  \]

- Engineering Strain
  \[
  \varepsilon = \begin{bmatrix}
  \lambda - 1 & 0 & 0 \\
  0 & \lambda^{-1/2} - 1 & 0 \\
  0 & 0 & \lambda^{-1/2} - 1
  \end{bmatrix}
  \]

- Difference
  \[
  E_{11} = \frac{1}{2}(\lambda^2 - 1), \quad e_{11} = \frac{1}{2}(1 - \lambda^{-2}), \quad \varepsilon_{11} = \lambda - 1
  \]
Polar Decomposition

• Want to separate deformation from rigid-body rotation
• Similar to principal directions of strain
• Unique decomposition of deformation gradient

\[ F = QU = VQ \]

- \( Q \): orthogonal tensor (rigid-body rotation)
- \( U, V \): right- and left-stretch tensor (symmetric)

• \( U \) and \( V \) have the same eigenvalues (principal stretches),
  but different eigenvectors

Polar Decomposition cont.

\[ dx = Q \cdot U \cdot dx = V \cdot Q \cdot dx \]

• Eigenvectors of \( U \): \( E_1, E_2, E_3 \)
• Eigenvectors of \( V \): \( e_1, e_2, e_3 \)
• Eigenvalues of \( U \) and \( V \): \( \lambda_1, \lambda_2, \lambda_3 \)
Polar Decomposition cont.

• Relation between \(U\) and \(\mathbf{C}\)

\[
U^2 = \mathbf{C} \quad U = \sqrt{\mathbf{C}}
\]

- \(U\) and \(\mathbf{C}\) have the same eigenvectors.
- Eigenvalue of \(U\) is the square root of that of \(\mathbf{C}\).

• How to calculate \(U\) from \(\mathbf{C}\)?

• Let eigenvectors of \(\mathbf{C}\) be \(\Phi = [\mathbf{E}_1 \quad \mathbf{E}_2 \quad \mathbf{E}_3]\)

• Then, \(\Lambda = \Phi^T \mathbf{C} \Phi\) where

\[
\Lambda = \begin{bmatrix}
\lambda_1^2 & 0 & 0 \\
0 & \lambda_2^2 & 0 \\
0 & 0 & \lambda_3^2
\end{bmatrix}
\]

Deformation tensor in principal directions

Polar Decomposition cont.

• And \(U = \Phi \sqrt{\Lambda} \Phi^T\)

\[
\sqrt{\Lambda} = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix}
\]

• General Deformation

\[
dx = \mathbf{F} \mathbf{d}X + \mathbf{b} = Q \mathbf{U} \mathbf{d}X + \mathbf{b}
\]

1. Stretch in principal directions
2. Rigid-body rotation
3. Rigid-body translation

Useful formulas

\[
\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{E}_i \otimes \mathbf{E}_i
\]

\[
\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{E}_i \otimes \mathbf{E}_i
\]

\[
\mathbf{Q} = \sum_{i=1}^{3} \mathbf{e}_i \otimes \mathbf{E}_i
\]

\[
\mathbf{b} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{e}_i \otimes \mathbf{e}_i
\]

\[
\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i
\]

\[
\mathbf{F} = \sum_{i=1}^{3} \lambda_i \mathbf{e}_i \otimes \mathbf{E}_i
\]
**Generalized Lagrangian Strain**

- G-L strain is a special case of general form of Lagrangian strain tensors (Hill, 1968)

\[ E_m = \frac{1}{2m} (U^{2m} - 1) \]

**Example – Polar Decomposition**

- Simple shear problem

\[
\begin{align*}
    x_1 &= x_1 + k x_2 & k &= \frac{2}{\sqrt{3}} \\
    x_2 &= x_2 \\
    x_3 &= x_3
\end{align*}
\]

- Deformation gradient \( F = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \)

- Deformation tensor \( C = F^T F = \begin{bmatrix} 1 & k \\ k & k^2 + 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & \frac{7}{3} \end{bmatrix} \)

- Find eigenvalues and eigenvectors of \( C \)

\[ \lambda_1 = 3, \quad \lambda_2 = \frac{1}{3} \]

\[ E_1 = \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), \quad E_2 = \left( -\frac{\sqrt{3}}{2}, \frac{1}{2} \right) \]
Example – Polar Decomposition cont.

• In $E_1 - E_2$ coordinates $C' = \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1/3 \end{bmatrix}$

• Principal Direction Matrix $\Phi = [E_1 \ E_2] = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

• Deformation tensor in principal directions
  $$\Lambda = \Phi^T \cdot C \cdot \Phi$$

• Stretch tensor
  $$\sqrt{\Lambda} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \end{bmatrix}$$
  $$U = \Phi \cdot \sqrt{\Lambda} \cdot \Phi^T = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & 5/2\sqrt{3} \end{bmatrix}$$

Example – Polar Decomposition cont.

• How $U$ deforms a square?
  $$U \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}, \quad U \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 5/2\sqrt{3} \end{bmatrix}$$

• Rotational Tensor
  $$Q = F \cdot U^{-1} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$
  $$Q \cdot \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Q \cdot \begin{bmatrix} 1/2 \\ 5/2\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1.15 \\ 1 \end{bmatrix}$$

  - $30^\circ$ clockwise rotation
  $$V = F \cdot Q^T = \begin{bmatrix} 5\sqrt{3}/6 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$
Example - Polar Decomposition cont.

• A straight line $X_2 = X_1 \tan \theta$ will deform to
  
  $X_1 = x_1 - kx_2, \quad X_2 = x_2$

  $\Rightarrow x_2 = (x_1 - kx_2) \tan \theta$

  $\Rightarrow x_1 = \left(\frac{1}{\tan \theta} + k\right)x_2$

• Consider a diagonal line: $\theta = 45^\circ$

  \[
  \tan \alpha = \frac{x_2}{x_1} = \frac{1}{1 + k} \quad \alpha = 24.9^\circ
  \]

• Consider a circle

  \[
  X_1^2 + X_2^2 = r^2
  \]

  \[
  (x_1 - kx_2)^2 + x_2^2 = r^2
  \]

  \[
  x_1^2 - 2kx_1x_2 + (1 + k^2)x_2^2 = r^2
  \]

  Equation of ellipse

Deformation of a Volume

• Infinitesimal volume by three vectors

  - Undeformed: $dV_0 = dX^1 \cdot (dX^2 \times dX^3) = e_{rst} dX_r^1 dX_s^2 dX_t^3$

  - Deformed: $dV_x = dX^1 \cdot (dX^2 \times dX^3) = e_{ijk} dX_i^1 dX_j^2 dX_k^3$

  $dV_x = e_{ijk} dX_i^1 dX_j^2 dX_k^3$

  $= e_{ijk} \left(\frac{\partial X_i}{\partial X_r} dX_r^1\right) \left(\frac{\partial X_j}{\partial X_s} dX_s^2\right) \left(\frac{\partial X_k}{\partial X_t} dX_t^3\right)$

  $= e_{ijk} \frac{\partial X_i}{\partial X_r} \frac{\partial X_j}{\partial X_s} \frac{\partial X_k}{\partial X_t} dX_r^1 dX_s^2 dX_t^3$

  $= e_{rst} J \quad dX_r^1 dX_s^2 dX_t^3$

  $= J dV_0$

  From Continuum Mechanics

  $J = \det F = \lambda_1 \lambda_2 \lambda_3$

  $e_{ijk} a_{ir} a_{js} a_{kt} = e_{rst} \det a$
Deformation of a Volume cont.

- Volume change

\[ dV_x = J \, dV_0 \]

- Volumetric Strain

\[ \frac{dV_x - dV_0}{dV_0} = J - 1 \]

- Incompressible condition: \( J = 1 \)

- Transformation of integral domain

\[
\iiint_{\Omega_x} f \, d\Omega = \iiint_{\Omega_0} f J \, d\Omega
\]

Example - Uniaxial Deformation of a Beam

- Initial dimension of \( L_0 \times h_0 \times h_0 \) deforms to \( L \times h \times h \)

\[
x_1 = \lambda_1 x_1 \quad \lambda_1 = \frac{L}{L_0} \\
x_2 = \lambda_2 x_2 \quad \lambda_2 = \frac{h}{h_0} \\
x_3 = \lambda_3 x_3 \quad \lambda_3 = \frac{h}{h_0}
\]

- Deformation gradient

\[
F = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \\
J = \det F = \lambda_1 \lambda_2 \lambda_3
\]

\[
J = \frac{L}{L_0} \left( \frac{h}{h_0} \right)^2 = \frac{LA}{L_0 A_0}
\]

- Constant volume

\[
J = 1 \implies h = h_0 \sqrt{\frac{L_0}{L}} \\
A = A_0 \frac{L_0}{L}
\]
Deformation of an Area

• Relationship between $dS_0$ and $dS_x$

$$NdS_0 = dX^1 \times dX^2 \quad NdS_0 = e_{ijk} dX_j^2 dX_k^2$$

$$ndS_x = dx^1 \times dx^2 \quad n_r dS_x = e_{rst} dx_s^1 dx_t^2$$

$$NdS_0 = e_{ijk} \frac{\partial X_j}{\partial x_s} \frac{\partial X_k}{\partial x_t} dx^2_s dx^2_t$$

$$n \frac{\partial X_j}{\partial x_r} \rightarrow \frac{\partial X_i}{\partial x_r} NdS_0 = e_{ijk} \frac{\partial X_j}{\partial x_r} \frac{\partial X_k}{\partial x_r} dx^2_s dx^2_t$$

Deformation of an Area cont..

• Results from Continuum Mechanics

$$e_{ijk} |F| = e_{rst} \frac{\partial X_r}{\partial x_i} \frac{\partial X_s}{\partial x_j} \frac{\partial X_t}{\partial x_k}$$

$$e_{rst} |F^{-1}| = e_{ijk} \frac{\partial X_i}{\partial x_r} \frac{\partial X_j}{\partial x_s} \frac{\partial X_k}{\partial x_t}.$$

• Use the second relation:

$$\frac{\partial X_i}{\partial x_r} NdS_0 = e_{ijk} \frac{\partial X_j}{\partial x_r} \frac{\partial X_k}{\partial x_r} dx^2_s dx^2_t = e_{rst} |F|^{-1} dx^2_s dx^2_t$$

$$ndS_x = JF^{-T} \cdot NdS_0 \quad \quad n \parallel F^{-T} \cdot N \quad \Rightarrow \quad n = \frac{F^{-T} \cdot N}{\| F^{-T} \cdot N \|}$$

$$dS_x = J \| F(x)^{-T} N(X) \| dS_0$$
**Stress Measures**

- Stress and strain (tensor) depend on the configuration
- Cauchy (True) Stress: Force acts on the deformed config.
  - Stress vector at $\Omega_x$:
    \[
    \mathbf{t} = \lim_{\Delta S_x \to 0} \frac{\Delta \mathbf{f}}{\Delta S_x} = \sigma \mathbf{n}
    \]
    **Cauchy Stress, sym**
  
  - Cauchy stress refers to the current deformed configuration as a reference for both area and force (**true stress**)

**Stress Measures cont.**

- The same force, but different area (undeformed area)
  
  \[
  \mathbf{T} = \lim_{\Delta S_0 \to 0} \frac{\Delta \mathbf{f}}{\Delta S_0} = \mathbf{P}^T \mathbf{N}
  \]
  **First Piola-Kirchhoff Stress**
  Not symmetric

- $\mathbf{P}$ refers to the force in the deformed configuration and the area in the undeformed configuration

- Make both force and area to refer to undeformed config.
  
  \[
  d\mathbf{f} = \sigma \mathbf{n} dS_x = \mathbf{P}^T \mathbf{N} dS_0 \quad \iff \quad \mathbf{n} dS_x = \mathbf{J} \mathbf{F}^{-T} \cdot \mathbf{N} dS_0
  \]

  \[
  d\mathbf{f} = \sigma (\mathbf{J} \mathbf{F}^{-T} \mathbf{N} dS_0) = \mathbf{P}^T \mathbf{N} dS_0
  \]

  \[
  \mathbf{P} = \mathbf{J} \mathbf{F}^{-1} \sigma \quad : \text{Relation between } \sigma \text{ and } \mathbf{P}
  \]
Stress Measures cont.

• Unsymmetric property of \( P \) makes it difficult to use
  - Remember we used the symmetric property of stress & strain several times in linear problems

• Make \( P \) symmetric by multiplying with \( F^{-T} \)

\[
S = P \cdot F^{-T} = JF^{-1} \cdot \sigma \cdot F^{-T}
\]

\[\sigma = \frac{1}{J} F \cdot S \cdot F^T\]

- Just convenient mathematical quantities

• Further simplification is possible by handling \( J \) differently

\[
\tau = J \sigma = F \cdot S \cdot F^T
\]

Kirchhoff Stress, symmetric

Stress Measures cont.

• Example

\[
\iiint_{\Omega_x} \sigma : \bar{\varepsilon} \, d\Omega_x = \iiint_{\Omega_0} \sigma : \bar{\varepsilon} J \, d\Omega_0 = \iiint_{\Omega_0} \tau : \bar{\varepsilon} \, d\Omega_0
\]

Integration can be done in \( \Omega_0 \)

• Observation
  - For linear problems (small deformation): \( \varepsilon \approx E \approx e \)
  - For linear problems (small deformation): \( \sigma \approx \tau \approx P \approx S \)
  - \( S \) and \( E \) are conjugate in energy
  - \( S \) and \( E \) are invariant in rigid-body motion
Example – Uniaxial Tension

- **Cauchy (true) stress**: \( \sigma_{11} = \frac{F}{A} \), \( \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{23} = \sigma_{13} = 0 \)

- **Deformation gradient**:

\[
F^{-1} = \begin{bmatrix}
\lambda_1^{-1} & 0 & 0 \\
0 & \lambda_2^{-1} & 0 \\
0 & 0 & \lambda_3^{-1}
\end{bmatrix}, \quad J = 1
\]

- **First P-K stress**

\[
p_{11} = (JF^{-1}\sigma)_{11} = \frac{F}{A} \frac{1}{\lambda_1} = \frac{F}{A} \frac{A}{A_0} = \frac{F}{A_0}
\]

- **Second P-K stress**

\[
s_{11} = (JF^{-1} \cdot \sigma \cdot F^{-T})_{11} = \frac{F}{A} \frac{1}{\lambda_1} = \frac{F}{A} \frac{A^2}{A_0^2} = \frac{FA}{A_0^2} = \frac{F}{A_0 \lambda_1}
\]

No clear physical meaning

---

**Summary**

- Nonlinear elastic problems use different measures of stress and strain due to changes in the reference frame.

- Lagrangian strain is independent of rigid-body rotation, but engineering strain is not.

- Any deformation can be uniquely decomposed into rigid-body rotation and stretch.

- The determinant of deformation gradient is related to the volume change, while the deformation gradient and surface normal are related to the area change.

- Four different stress measures are defined based on the reference frame.

- All stress and strain measures are identical when the deformation is infinitesimal.
3.3
Nonlinear Elastic Analysis

Goals

• Understanding the principle of minimum potential energy
  - Understand the concept of variation
• Understanding St. Venant-Kirchhoff material
• How to obtain the governing equation for nonlinear elastic problem
• What is the total Lagrangian formulation?
• What is the updated Lagrangian formulation?
• Understanding the linearization process
**Numerical Methods for Nonlinear Elastic Problem**

- We will obtain the variational equation using the **principle of minimum potential energy**
  - Only possible for elastic materials (potential exists)
- The N-R method will be used (need Jacobian matrix)
- **Total Lagrangian (material) formulation** uses the undeformed configuration as a reference, while the **updated Lagrangian (spatial)** uses the current configuration as a reference
- The total and updated Lagrangian formulations are **mathematically equivalent** but have different aspects in computation

**Total Lagrangian Formulation**

- Using **incremental force method** and **N-R method**
  - Total No. of load steps \( N \), current load step \( n \)
  \[
  f^{n+1} = f^n + \Delta f^n
  \]
- Assume that the solution has converged up to \( t_n \)
- Want to find the equilibrium state at \( t_{n+1} \)
Total Lagrangian Formulation cont.

- In TL, the **undeformed configuration** is the reference
- 2nd P-K stress ($S$) and G-L strain ($E$) are the natural choice
- In elastic material, **strain energy density $W$** exists, such that
  \[
  \text{stress} = \frac{\partial W}{\partial \text{strain}}
  \]
- We need to express $W$ in terms of $E$

Strain Energy Density and Stress Measures

- By differentiating strain energy density with respect to proper strains, we can obtain stresses
- When $W(E)$ is given
  \[
  S = \frac{\partial W(E)}{\partial E} \quad \text{Second P-K stress}
  \]
- When $W(F)$ is given
  \[
  \frac{\partial W}{\partial F} = \frac{\partial W}{\partial E} : \frac{\partial E}{\partial F} = F \cdot \frac{\partial W}{\partial E} = F \cdot S = P^T \quad \text{First P-K stress}
  \]
- It is difficult to have $W(\varepsilon)$ because $\varepsilon$ depends on rigid-body rotation. Instead, we will use **invariants** in Section 3.5
St. Venant-Kirchhoff Material

- Strain energy density for St. Venant-Kirchhoff material
  \[ W(E) = \frac{1}{2} E : D : E \]
  
  Contraction operator: \( a : b = a_{ij}b_{ij} \)

- Fourth-order constitutive tensor (isotropic material)
  \[ D = \lambda 1 \otimes 1 + 2\mu I \]
  
  - Lame's constants:
    \[ \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)} \]
  
  - Identity tensor (2\textsuperscript{nd} order): \( 1 = [\delta_{ij}] \)
  
  - Identity tensor (4\textsuperscript{th} order):
    \[ I_{ijkl} = \frac{1}{2} (\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{jk}) \]
  
  \[ I : a = a, \quad \forall \text{2nd-order sym. } a \]
  
  \[ 1 : a = \text{tr}(a) = a_{ii} = a_{11} + a_{22} + a_{33} \]
  
  - Tensor product:
    \[ a \otimes a = a_{ij}a_{kl} \text{ (4th-order)} \]

St. Venant-Kirchhoff Material cont.

- Stress calculation
  
  differentiate strain energy density
  \[ S = \frac{\partial W(E)}{\partial E} = D : E = \lambda \text{tr}(E)1 + 2\mu E \]

  - Limited to small strain but large rotation
    \[ E = \frac{1}{2}(F^TF - 1) = \frac{1}{2}(U^TQ^TQu - 1) = \frac{1}{2}(U^2 - 1) \]

  - Rigid-body rotation is removed and only the stretch tensor contributes to the strain
  
  - Can show
    \[ S = \frac{\partial W}{\partial E} = 2 \frac{\partial W}{\partial C} \]
    
    Deformation tensor
Example

- E = 30,000 and ν = 0.3
- G-L strain: \[ E = \begin{bmatrix} 0.389 & 0 \\ 0 & -0.255 \end{bmatrix} \]
- Lame’s constants:
  \[ \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} = 17,308 \quad \mu = \frac{E}{2(1 + \nu)} = 11,538 \]
- 2nd P-K Stress:
  \[ S = \lambda \text{tr}(E)\mathbf{1} + 2\mu E = \lambda (.389 - .255) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\mu \begin{bmatrix} .389 & 0 \\ 0 & - .255 \end{bmatrix} \]
  \[ = \begin{bmatrix} 11,296 & 0 \\ 0 & -3,565 \end{bmatrix} \]
  \[ \sigma = \frac{1}{J} F S F^T = \begin{bmatrix} -1,872 & 0 \\ 0 & 21,516 \end{bmatrix} \]

Example - Simple Shear Problem

- Deformation map
  \[ x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3 \]
  \[ F = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \]
  \[ E = \frac{1}{2} (F^T F - \mathbf{1}) = \frac{1}{2} \begin{bmatrix} 0 & k \\ k & k^2 \end{bmatrix} \]
- Material properties
  \[ \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} = 40\text{MPa} \quad \mu = \frac{E}{2(1 + \nu)} = 40\text{MPa} \]
- 2nd P-K stress
  \[ S = \lambda \text{tr}(E)\mathbf{1} + 2\mu E = 20 \begin{bmatrix} k^2 & 2k \\ 2k & 3k^2 \end{bmatrix} \text{MPa} \]
  \[ \sigma = \frac{1}{J} F S F^T = 20 \begin{bmatrix} 5k^2 + 3k^4 & 2k + 3k^3 \\ 2k + 3k^3 & 3k^2 \end{bmatrix} \text{MPa} \]
Boundary Conditions

- **Boundary Conditions**
  \[ u = g, \quad \text{on } \Gamma^h \quad \text{Essential (displacement) boundary} \]
  \[ t = P^T N, \quad \text{on } \Gamma^s \quad \text{Natural (traction) boundary} \]
  \[
  \text{You can't use } S
  \]

- **Solution space (set)**
  \[ \mathcal{V} = \{ u | u \in [H^1(\Omega)]^3, \ u|_{\Gamma^h} = g \} \]

- **Kinematically admissible space**
  \[ \mathcal{Z} = \{ \overline{u} | \overline{u} \in [H^1(\Omega)]^3, \ \overline{u}|_{\Gamma^h} = 0 \} \]

Variational Formulation

- **We want to minimize the potential energy (equilibrium)**
  \[ \Pi^\text{int}: \text{stored internal energy} \]
  \[ \Pi^\text{ext}: \text{potential energy of applied loads} \]
  \[ \Pi(u) = \Pi^\text{int}(u) + \Pi^\text{ext}(u) \]
  \[ = \int_{\Omega_0} W(E) \, d\Omega - \int_{\Omega_0} u^T f^b \, d\Omega - \int_{\Gamma^s} u^T t \, d\Gamma \]

- **Want to find** \( u \in \mathcal{V} \) **that minimizes the potential energy**
  - Perturb \( u \) in the direction of \( \overline{u} \in \mathcal{Z} \) proportional to \( \tau \)
    \[ u_t = u + \tau \overline{u} \]
  - If \( u \) minimizes the potential, \( \Pi(u) \) must be smaller than \( \Pi(u_t) \) for all possible \( \overline{u} \)
Variational Formulation cont.

- **Variation of Potential Energy** (Directional Derivative)
  \[ \Pi(u, \bar{u}) = \frac{d}{d\tau} \Pi(u + \tau \bar{u}) \bigg|_{\tau=0} \]
  We will use “over-bar” for variation

  - \( \Pi \) depends on \( u \) only, but \( \Pi \) depends on both \( u \) and \( \bar{u} \)
  - **Minimum potential energy** happens when its variation becomes zero for every possible \( \bar{u} \)
  - One-dimensional example

\[ \begin{align*}
\Pi(u) & \rightarrow \text{At minimum, all directional derivatives are zero}
\end{align*} \]

**Example – Linear Spring**

- Potential energy: \( \Pi(u) = \frac{1}{2} k \cdot u^2 - f \cdot u \)
- Perturbation: \( \Pi(u + \tau \bar{u}) = \frac{1}{2} k \cdot (u + \tau \bar{u})^2 - f \cdot (u + \tau \bar{u}) \)
- Differentiation: \( \frac{d}{d\tau}[\Pi(u + \tau \bar{u})] = k \cdot (u + \tau \bar{u}) \cdot \bar{u} - f \cdot \bar{u} \)
- Evaluate at original state:
  \[ \frac{d}{d\tau}[\Pi(u + \tau \bar{u})]_{\tau=0} = k \cdot u \cdot \bar{u} - f \cdot \bar{u} = 0 \]

**Variation is similar to differentiation !!!**
Variational Formulation cont.

• Variational Equation

\[ \Pi(u, \bar{u}) = \iint_{\Omega_0} \frac{\partial W(E)}{\partial E} : \bar{E} \, d\Omega = \iint_{\Omega_0} \bar{u}^T f_b \, d\Omega - \int_{\Gamma_o} \bar{u}^T t \, d\Gamma = 0 \]

- From the definition of stress

\[ \iint_{\Omega_0} S : \bar{E} \, d\Omega = \iint_{\Omega_0} \bar{u}^T f_b \, d\Omega + \int_{\Gamma_o} \bar{u}^T t \, d\Gamma \]

Variational equation in TL formulation

- Note: load term is similar to linear problems

- Nonlinearity in the strain energy term

• Need to write LHS in terms of \( u \) and \( \bar{u} \)

Variational Formulation cont.

• How to express strain variation

\[ E(u) = \frac{1}{2}(C - I) = \frac{1}{2} \left( \nabla_0 u + \nabla_0 u^T + \nabla_0 u^T \nabla_0 u \right) \]

\[ \bar{E}(u, \bar{u}) = \frac{d}{d\tau} E(u + \tau \bar{u}) \bigg|_{\tau=0} \]

\[ = \frac{1}{2} \left( \nabla_0 \bar{u} + \nabla_0 \bar{u}^T + \nabla_0 \bar{u}^T \nabla_0 u + \nabla_0 u^T \nabla_0 \bar{u} \right) \]

\[ = \frac{1}{2} \left( (I + \nabla_0 u^T) \nabla_0 \bar{u} + \nabla_0 \bar{u}^T (I + \nabla_0 u) \right) \]

\[ = \frac{1}{2} \left( F^T \nabla_0 \bar{u} + \nabla_0 \bar{u}^T F \right) \]

\[ \bar{E}(u, \bar{u}) = \text{sym}(\nabla_0 \bar{u}^T F) \]

Note: \( E(u) \) is nonlinear, but \( \bar{E}(u, \bar{u}) \) is linear
Variational Formulation cont.

- Variational Equation

\[
\iint_{\Omega_0} s : \bar{E} \, d\Omega = \int_{\Omega_0} \bar{u}^T f^b \, d\Omega + \int_{\Gamma^s} \bar{u}^T t \, d\Gamma \quad \text{for all } \bar{u} \in \mathbb{Z}
\]

\[
a(u, \bar{u}) = \ell(\bar{u}), \quad \forall \bar{u} \in \mathbb{Z}
\]

- Linear in terms of strain if St. Venant-Kirchhoff material is used
- Also linear in terms of \( \bar{u} \)
- Nonlinear in terms of \( u \) because displacement-strain relation is nonlinear

Linearization (Increment)

- Linearization process is similar to variation and/or differentiation
  - First-order Taylor series expansion
  - Essential part of Newton-Raphson method
- Let \( f(x^{k+1}) = f(x^k + \Delta u^k) \), where we know \( x^k \) and want to calculate \( \Delta u^k \)

\[
f(x^{k+1}) = f(x^k) + \frac{df(x^k)}{dx} \cdot \Delta u^k + \text{H.O.T.}
\]

- The first-order derivative is indeed linearization of \( f(x) \)

\[
L[f] \equiv \frac{d}{d\omega} f(x + \omega \Delta u) \bigg|_{\omega=0} = \frac{\partial f}{\partial x} \cdot \Delta u
\]

\[
\delta f = \bar{f} \equiv \frac{d}{d\tau} f(x + \tau \bar{u}) \bigg|_{\tau=0} = \frac{\partial f}{\partial x} \cdot \bar{u}
\]
Linearization of Residual

- We are still in continuum domain (not discretized yet)
- Residual \( R(u) = a(u, \ddot{u}) - \ell(\ddot{u}) \)
- We want to linearize \( R(u) \) in the direction of \( \Delta u \)
  - First, assume that \( u \) is perturbed in the direction of \( \Delta u \) using a variable \( \tau \). Then linearization becomes

\[
L[R(u)] = \left. \frac{\partial R(u + \tau \Delta u)}{\partial \tau} \right|_{\tau=0} = \left[ \frac{\partial R}{\partial u} \right]^T \Delta u
\]

- \( R(u) \) is nonlinear w.r.t. \( u \), but \( L[R(u)] \) is linear w.r.t. \( \Delta u \)
- Iteration \( k \) did not converged, and we want to make the residual at iteration \( k+1 \) zero

\[
R(u^{k+1}) = \left[ \frac{\partial R(u^k)}{\partial u} \right]^T \Delta u^k + R(u^k) = 0
\]

Newton-Raphson Iteration by Linearization

- This is N-R method (see Chapter 2)

\[
\left[ \frac{\partial R(u^k)}{\partial u} \right]^T \Delta u^k = -R(u^k)
\]

- Update state \( u^{k+1} = u^k + \Delta u^k \)
  \( x^{k+1} = X + u^{k+1} \)

- We know how to calculate \( R(u^k) \), but how about \( \left[ \frac{\partial R(u^k)}{\partial u} \right] \)?

\[
\frac{\partial}{\partial u} [R(u)] = \frac{\partial}{\partial u} [a(u, \ddot{u}) - \ell(\ddot{u})]
\]

- Only linearization of energy form will be required
- We will address displacement-dependent load later
Linearization cont.

- Linearization of energy form

\[ L[a(u, \bar{u})] = L \left[ \int_\Omega S : \bar{E} \, d\Omega \right] = \int_\Omega \left[ \Delta S : \bar{E} + S : \Delta \bar{E} \right] d\Omega \]

  - Note that the domain is fixed (undeformed reference)
  - Need to express in terms of displacement increment \( \Delta u \)

- Stress increment (St. Venant-Kirchhoff material)

\[ \Delta S = \frac{\partial S}{\partial \bar{E}} : \Delta \bar{E} = D : \Delta \bar{E} \]

- Strain increment (Green-Lagrange strain)

\[ \Delta E = \frac{1}{2} (\Delta F^T F + F^T \Delta F) \]

\[ \Delta F = \Delta \left( \frac{\partial \bar{X}}{\partial \bar{X}} \right) = \Delta \left( \frac{\partial (X + u)}{\partial X} \right) = \frac{\partial \Delta u}{\partial X} = \nabla_0 \Delta u \]

Linearization cont.

- Strain increment

\[ \Delta E = \frac{1}{2} (\Delta F^T F + F^T \Delta F) \]

\[ = \frac{1}{2} (\nabla_0 \Delta u^T F + F^T \nabla_0 \Delta u) \]

\[ = \text{sym}(\nabla_0 \Delta u^T F) \quad \text{!!! Linear w.r.t. } \Delta u \]

- Inc. strain variation

\[ \Delta \bar{E} = \Delta [\text{sym}(\nabla_0 \bar{u}^T F)] \]

\[ = \text{sym}(\nabla_0 \bar{u}^T \Delta F) \]

\[ = \text{sym}(\nabla_0 \bar{u}^T \nabla_0 \Delta u) \quad \text{!!! Linear w.r.t. } \Delta u \]

- Linearized energy form

\[ L[a(u, \bar{u})] = \int_\Omega [\bar{E} : D : \Delta \bar{E} + S : \Delta \bar{E}] d\Omega = a^*(u; \Delta u, \bar{u}) \]

  - Implicitly depends on \( u \), but bilinear w.r.t. \( \Delta u \) and \( \bar{u} \)
  - First term: tangent stiffness
  - Second term: initial stiffness
Linearization cont.

- N-R Iteration with Incremental Force
  - Let $t_n$ be the current load step and $(k+1)$ be the current iteration
  - Then, the N-R iteration can be done by
    \begin{equation}
    a^*(\mathbf{u}^k; \Delta \mathbf{u}^k, \bar{u}) = \ell(\bar{u}) - a(\mathbf{u}^k, \bar{u}), \quad \forall \bar{u} \in \mathbb{R}
    \end{equation}
  - Update the total displacement
    \[ n\mathbf{u}^{k+1} = n\mathbf{u}^k + \Delta \mathbf{u}^k \]

- In discrete form
  \[ \{\bar{d}\}^T [n\mathbf{K}_n^k] \{\Delta d^k\} = \{\bar{d}\}^T \{n\mathbf{R}^k\} \]

- What are $[n\mathbf{K}_n^k]$ and $\{n\mathbf{R}^k\}$?

Example - Uniaxial Bar

- Kinematics
  \[ \frac{du}{dX} = u_2, \quad \frac{d\bar{u}}{dX} = \bar{u}_2 \]
  \[ E_{11} = \frac{du}{dX} + \frac{1}{2} \left( \frac{du}{dX} \right)^2 = u_2 + \frac{1}{2} (u_2)^2 \]

- Strain variation
  \[ \bar{E}_{11} = \frac{d\bar{u}}{dX} + \frac{du}{dX} \frac{d\bar{u}}{dX} = \bar{u}_2 (1 + u_2) \]

- Strain energy density and stress
  \[ W(E_{11}) = \frac{1}{2} E \cdot (E_{11})^2 \quad S_{11} = \frac{\partial W}{\partial E_{11}} = E \cdot E_{11} = E \left( u_2 + \frac{1}{2} (u_2)^2 \right) \]

- Energy and load forms
  \[ a(u, \bar{u}) = \int_0^{L_0} S_{11} \bar{E}_{11} A dX = S_{11} A L_0 (1 + u_2) \bar{u}_2 \quad \ell(\bar{u}) = \bar{u}_2 F \]

- Variational equation
  \[ R = \bar{u}_2 \left( S_{11} A L_0 (1 + u_2) - F \right) = 0, \quad \forall \bar{u}_2 \]
Example - Uniaxial Bar

- **Linearization**

\[ \Delta S_{11} = E \Delta E_{11} = E(1 + u_2) \Delta u_2 \quad \Delta \bar{E}_{11} = \bar{u}_2 \Delta u_2 \]

\[ a^*(u; \Delta u, \bar{u}) = \int_0^l \left( \bar{E}_{11} \cdot E \cdot \Delta E_{11} + S_{11} \cdot \Delta \bar{E}_{11} \right) dX \]

\[ = EAL_0 (1 + u_2)^2 \bar{u}_2 \Delta u_2 + S_{11} AL_0 \bar{u}_2 \Delta u_2 \]

- **N-R iteration**

\[ [E(1 + u_2^k)^2 + S_{11}^k] AL_0 \Delta u_2^k = F - S_{11}^k (1 + u_2^k) AL_0 \]

\[ u_2^{k+1} = u_2^k + \Delta u_2^k \]

---

### Example - Uniaxial Bar

(a) with initial stiffness

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<thead>
<tr>
<th>Iteration</th>
<th>( u )</th>
<th>Strain</th>
<th>Stress</th>
<th>conv</th>
</tr>
</thead>
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<td>0.0000</td>
<td>0.0000</td>
<td>9.999E−01</td>
</tr>
<tr>
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<td>4.236E−06</td>
</tr>
</tbody>
</table>

(b) without initial stiffness

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<tr>
<th>Iteration</th>
<th>( u )</th>
<th>Strain</th>
<th>Stress</th>
<th>conv</th>
</tr>
</thead>
<tbody>
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<td>0.0000</td>
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</tbody>
</table>
Updated Lagrangian Formulation

- The current configuration is the reference frame
  - Remember it is unknown until we solve the problem
  - How are we going to integrate if we don't know integral domain?

- **What stress and strain should be used?**
  - For stress, we can use Cauchy stress ($\sigma$)
  - For strain, engineering strain is a pair of Cauchy stress
  - But, it must be defined in the current configuration

$$\varepsilon = \frac{1}{2} \left( \frac{\partial \mathbf{u}^T}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right) = \text{sym}(\nabla \mathbf{x} \mathbf{u})$$

Variational Equation in UL

- Instead of deriving a new variational equation, we will convert from TL equation

$$\sigma = \frac{1}{J} \mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^T$$

$$\Rightarrow \, \mathbf{S} = J \mathbf{F}^{-1} \cdot \mathbf{\sigma} \cdot \mathbf{F}^{-T}$$

$$\begin{align*}
\bar{\mathbf{E}} &= \frac{1}{2} \left( \frac{\partial \mathbf{u}^T}{\partial \mathbf{x}} \mathbf{F} + \mathbf{F}^T \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right) \\
&= \frac{1}{2} \mathbf{F}^T \left( \mathbf{F}^{-T} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{F}^{-1} \right) \mathbf{F} \\
&= \frac{1}{2} \mathbf{F}^T \left( \frac{\partial \mathbf{X}}{\partial \mathbf{x}}^T \frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \frac{\partial \mathbf{X}}{\partial \mathbf{x}} \right) \mathbf{F} \\
&= \frac{1}{2} \mathbf{F}^T \left( \frac{\partial \mathbf{X}}{\partial \mathbf{x}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \frac{\partial \mathbf{X}}{\partial \mathbf{x}} \right) \mathbf{F} \\
&= \frac{1}{2} \mathbf{F}^T \left( \frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right) \mathbf{F} \\
&= \mathbf{F}^T \cdot \bar{\varepsilon} \cdot \mathbf{F}
\end{align*}$$
Variational Equation in UL cont.

- Energy Form

\[ a(u, \bar{u}) = \iiint_{\Omega} S : \bar{E} \, d\Omega = \iiint_{\Omega} (J F^{-1} \sigma F^{-T}) : (F^T \varepsilon F) \, d\Omega \]

\[ F_{ik}^{-1} \sigma_{kl} F_{jl}^{-1} F_{mj} \varepsilon_{mn} F_{nj} = \delta_{mk} \delta_{ni} \sigma_{kl} \varepsilon_{mn} = \sigma_{mn} \varepsilon_{mn} \]

\[ \iiint_{\Omega} S : \bar{E} \, d\Omega = \iiint_{\Omega} \sigma : \varepsilon \, J \, d\Omega = \iiint_{\Omega} \sigma : \varepsilon \, d\Omega \]

- We just showed that material and spatial forms are mathematically equivalent

- Although they are equivalent, we use different notation:

\[ a(u, \bar{u}) = \iiint_{\Omega} \sigma : \varepsilon \, d\Omega \]

Is this linear or nonlinear?

- Variational Equation

\[ a(u, \bar{u}) = \ell(\bar{u}), \quad \forall \bar{u} \in \mathbb{Z} \]

What happens to load form?

Linearization of UL

- Linearization of \( a_x(u, \bar{u}) \) will be challenging because we don’t know the current configuration (it is function of \( u \))

- Similar to the energy form, we can convert the linearized energy form of TL

- Remember \( a^*(u; \Delta u, \bar{u}) = \iiint_{\Omega} [\bar{E} : \mathcal{D} : \Delta \mathcal{E} + \mathcal{S} : \Delta \bar{E}] d^0 \Omega \)

- Initial stiffness term

\[ \mathcal{S} : \Delta \bar{E} = J (F^{-1} \sigma F^{-T}) : \frac{1}{2} \left( \frac{\partial \bar{u}^T}{\partial X} \frac{\partial \Delta u}{\partial X} + \frac{\partial \Delta u^T}{\partial X} \frac{\partial \bar{u}}{\partial X} \right) \]

\[ = J F_{ik}^{-1} \sigma_{kl} F_{jl}^{-1} \frac{1}{2} \left( \frac{\partial \bar{u}_m}{\partial X_i} \frac{\partial \Delta u_m}{\partial X_j} + \frac{\partial \Delta u_m}{\partial X_i} \frac{\partial \bar{u}_m}{\partial X_j} \right) \]

\[ = J \sigma_{kl} \frac{1}{2} \left( \frac{\partial \bar{u}_m}{\partial X_k} \frac{\partial \Delta u_m}{\partial X_l} + \frac{\partial \Delta u_m}{\partial X_k} \frac{\partial \bar{u}_m}{\partial X_l} \right) \rightarrow \eta_{kl}(\Delta u, \bar{u}) \]
Linearization of UL cont.

- Initial stiffness term

\[ S : \Delta \bar{E} = J \sigma : \eta(\Delta u, \bar{u}) \]
\[ \eta(\Delta u, \bar{u}) = \text{sym}(\nabla_x \bar{u}^T \nabla_x \Delta u) \]

- Tangent stiffness term

\[ (\bar{E} : D : \Delta \bar{E}) = (F^T \cdot \bar{\varepsilon} \cdot F) : D : (F^T \cdot \Delta \varepsilon \cdot F) \]
\[ = F_{ki} \bar{\varepsilon}_{kl} F_{lj} D_{ijmn} F_{pm} \Delta \varepsilon_{pq} F_{qn} \]
\[ = J \bar{\varepsilon}_{kl} \begin{bmatrix} 1 \\ \frac{1}{J} \end{bmatrix} F_{ki} F_{lj} D_{ijmn} F_{pm} F_{qn} \Delta \varepsilon_{pq} \]

\[ \bar{E} : D : \Delta \bar{E} = J \bar{\varepsilon} : c : \Delta \varepsilon \]

where
\[ c_{ijkl} = \frac{1}{J} F_{ir} F_{js} F_{km} F_{ln} D_{rs} \]

Spatial Constitutive Tensor

- For St. Venant-Kirchhoff material

\[ D = \lambda (1 \otimes 1) + 2 \mu I \]
\[ D_{rs} = \lambda \delta_{rs} \delta_{mn} + \mu (\delta_{rm} \delta_{sn} + \delta_{rn} \delta_{sm}) \]

- It is possible to show

\[ c_{ijkl} = \frac{1}{J} \left[ \lambda b_{ij} b_{kl} + \mu (b_{ik} b_{jl} + b_{il} b_{jk}) \right] . \]

- Observation
  - D (material) is constant, but c (spatial) is not
  - S = D : E, \quad \sigma \neq c : \varepsilon
Linearization of UL cont.

- From equivalence, the energy form is linearized in TL and converted to UL

\[
L[a(u, \bar{u})] = \int \int_{\Omega_0} [\varepsilon : c : \Delta \varepsilon + \sigma : \eta] J \, d\Omega
\]

\[
a^*(u; \Delta u, \bar{u}) = \int \int_{\Omega_x} [\varepsilon : c : \Delta \varepsilon + \sigma : \eta] \, d\Omega
\]

- N-R Iteration

\[
a^*(n^k u^k; \Delta u^k, \bar{u}) = \ell(\bar{u}) - a(n^k u^k, \bar{u}), \quad \forall \bar{u} \in \mathbb{Z}
\]

- Observations
  - Two formulations are theoretically identical with different expression
  - Numerical implementation will be different
  - Different constitutive relation

Example – Uniaxial Bar

- Kinematics

\[
\frac{du}{dx} = \frac{u_2}{1 + u_2}, \quad \frac{d\bar{u}}{dx} = \frac{\bar{u}_2}{1 + u_2}
\]

- Deformation gradient: \( F_{11} = \frac{dx}{dX} = 1 + u_2, \quad J = 1 + u_2 \)

- Cauchy stress: \( \sigma_{11} = \frac{1}{J} F_{11} S_{11} F_{11}^{-1} = E(u_2 + \frac{1}{2} u_2^2)(1 + u_2) \)

- Strain variation: \( \varepsilon_{11}(\bar{u}) = F_{11}^{-1} E_{11} F_{11}^{-1} = \frac{\bar{u}_2}{1 + u_2} \)

- Energy & load forms: \( a(u, \bar{u}) = \int_0^L \sigma_{11} \varepsilon_{11}(\bar{u}) A \, dx = \sigma_{11} A \bar{u}_2 \quad \ell(\bar{u}) = \bar{u}_2 F \)

- Residual: \( R = \bar{u}_2 (\sigma_{11} A - F) = 0, \quad \forall \bar{u}_2 \)
Example - Uniaxial Bar

- Spatial constitutive relation: \( c_{1111} = \frac{1}{J} F_{11} F_{11} F_{11} E = (1 + u_2)^3 E \)

- Linearization:
  \[
  \int_0^L \varepsilon_{11}(\bar{u})c_{1111}\varepsilon_{11}(\Delta u)A \, dx = EA(1 + u_2)^2 \bar{u}_2 \Delta u_2 \\
  \int_0^L \sigma_{11} \eta_{11}(\Delta u, \bar{u})A \, dx = \frac{\sigma_{11}A}{1 + u_2} \bar{u}_2 \Delta u_2 \\
  a^*(u; \Delta u, \bar{u}) = \int_0^L (\varepsilon_{11}(\bar{u})c_{1111}\varepsilon_{11}(\Delta u) + \sigma_{11} \eta(\Delta u, \bar{u}))A \, dx \\
  = EA(1 + u_2)^2 \bar{u}_2 \Delta u_2 + \frac{\sigma_{11}}{1 + u_2} A \bar{u}_2 \Delta u_2
  \]

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<th>conv</th>
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Section 3.5

Hyperelastic Material Model
Goals

• Understand the definition of hyperelastic material
• Understand strain energy density function and how to use it to obtain stress
• Understand the role of invariants in hyperelasticity
• Understand how to impose incompressibility
• Understand mixed formulation and perturbed Lagrangian formulation
• Understand linearization process when strain energy density is written in terms of invariants

What Is Hyperelasticity?

• Hyperelastic material - stress-strain relationship derives from a strain energy density function
  - Stress is a function of total strain (independent of history)
  - Depending on strain energy density, different names are used, such as Mooney-Rivlin, Ogden, Yeoh, or polynomial model
• Generally comes with incompressibility ($J = 1$)
  - The volume preserves during large deformation
  - Mixed formulation - completely incompressible hyperelasticity
  - Penalty formulation - nearly incompressible hyperelasticity
• Example: rubber, biological tissues
  - nonlinear elastic, isotropic, incompressible and generally independent of strain rate
• Hypoelastic material: relation is given in terms of stress and strain rates
Strain Energy Density

• We are interested in isotropic materials
  - Material frame indifference: no matter what coordinate system is chosen, the response of the material is identical
  - The components of a deformation tensor depends on coord. system
  - Three invariants of $C$ are independent of coord. system

• Invariants of $C$

\[
I_1 = \text{tr}(C) = C_{11} + C_{22} + C_{33} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2
\]

\[
I_2 = \frac{1}{2} \left[ (\text{tr}C)^2 - \text{tr}(C^2) \right] = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2
\]

\[
I_3 = \det C = \lambda_1^2 \lambda_2^2 \lambda_3^2
\]

- In order to be material frame indifferent, material properties must be expressed using invariants
- For incompressibility, $I_3 = 1$

Strain Energy Density cont.

• Strain Energy Density Function

\[
W(I_1, I_2, I_3) = \sum_{m+n+k=1}^{\infty} A_{mnk} (I_1 - 3)^m (I_2 - 3)^n (I_3 - 1)^k
\]

- For incompressible material

\[
W(I_1, I_2) = \sum_{m+n=1}^{\infty} A_{mn} (I_1 - 3)^m (I_2 - 3)^n
\]

- Ex: Neo-Hookean model

\[
W(I_1) = A_{10} (I_1 - 3)
\]

\[
A_{10} = \frac{\mu}{2}
\]

- Mooney-Rivlin model

\[
W(I_1, I_2) = A_{10} (I_1 - 3) + A_{01} (I_2 - 3)
\]
Strain Energy Density cont.

- **Strain Energy Density Function**
  - Yeoh model
    \[ W_1(I_1) = A_{10}(I_1 - 3) + A_{20}(I_1 - 3)^2 + A_{30}(I_1 - 3)^3 \]
  - Ogden model
    \[ W_1(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i} \left( \lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3 \right) \]
    \[ \mu = \frac{1}{2} \sum_{i=1}^{N} \alpha_i \mu_i \]
  - When \( N = 1 \) and \( \alpha_1 = 1 \), Neo-Hookean material
  - When \( N = 2 \), \( \alpha_1 = 2 \), and \( \alpha_2 = -2 \), Mooney-Rivlin material

Example – Neo-Hookean Model

- **Uniaxial tension with incompressibility**
  \[ \lambda_1 = \lambda \quad \lambda_2 = \lambda_3 = 1 / \sqrt{\lambda} \]
- **Energy density**
  \[ W = A_{10}(I_1 - 3) = A_{10}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) = A_{10}(\lambda^2 + \frac{2}{\lambda} - 3) \]
- **Nominal stress**
  \[ P = \frac{\partial W}{\partial \lambda} = 2A_{10} \left( \lambda - \frac{1}{\lambda^2} \right) = \mu \left( 1 + \varepsilon - \frac{1}{(1 + \varepsilon)^2} \right) \]
Example – St. Venant Kirchhoff Material

• Show that St. Venant-Kirchhoff material has the following strain energy density

\[ W(E) = \frac{\lambda}{2} \left[ \text{tr}(E) \right]^2 + \mu \text{tr}(E^2) \]

\[ S = \frac{\partial W(E)}{\partial E} = \lambda \text{tr}(E) \frac{\partial \text{tr}(E)}{\partial E} + \mu \frac{\partial \text{tr}(E^2)}{\partial E} \]

• First term

\[ \text{tr}(E) = 1 : E \quad \frac{\partial \text{tr}(E)}{\partial E} = 1 \]

\[ \lambda \text{tr}(E) \frac{\partial \text{tr}(E)}{\partial E} = \lambda (1 : E) = \lambda (1 \otimes 1) : E \]

• Second term

\[ \frac{\partial E_{ij} E_{ji}}{\partial E_{kl}} = \delta_{ik} \delta_{jl} E_{ji} + E_{ij} \delta_{jk} \delta_{il} = E_{lk} + E_{lk} = 2E_{lk} \]

Example – St. Venant Kirchhoff Material cont.

• Therefore

\[ S = \lambda \text{tr}(E) \frac{\partial \text{tr}(E)}{\partial E} + \mu \frac{\partial \text{tr}(E^2)}{\partial E} \]

\[ = \lambda (1 \otimes 1) : E + 2\mu E \]

\[ = \left[ \lambda (1 \otimes 1) + 2\mu I \right] : E \]

\[ D \]
Nearly Incompressible Hyperelasticity

- Incompressible material
  - Cannot calculate stress from strain. Why?

- Nearly incompressible material
  - Many material show nearly incompressible behavior
  - We can use the bulk modulus to model it

- Using $I_1$ and $I_2$ enough for incompressibility?
  - No, $I_1$ and $I_2$ actually vary under hydrostatic deformation
  - We will use reduced invariants: $J_1$, $J_2$, and $J_3$

$$J_1 = I_1 I_3^{-1/3} \quad J_2 = I_2 I_3^{-2/3} \quad J_3 = J = I_3^{1/2}$$

- Will $J_1$ and $J_2$ be constant under dilatation?

Locking

- What is locking
  - Elements do not want to deform even if forces are applied
  - Locking is one of the most common modes of failure in NL analysis
  - It is very difficult to find and solutions show strange behaviors

- Types of locking
  - Shear locking: shell or beam elements under transverse loading
  - Volumetric locking: large elastic and plastic deformation

- Why does locking occur?
  - Incompressible sphere under hydrostatic pressure

[Diagram of sphere with pressure applied and pressure-strain graph showing no unique pressure for given displacement]
How to solve locking problems?

• Mixed formulation (incompressibility)
  - Can’t interpolate pressure from displacements
  - Pressure should be considered as an independent variable
  - Becomes the Lagrange multiplier method
  - The stiffness matrix becomes positive semi-definite

Penalty Method

• Instead of incompressibility, the material is assumed to be nearly incompressible
• This is closer to actual observation
• Use a large bulk modulus (penalty parameter) so that a small volume change causes a large pressure change
• Large penalty term makes the stiffness matrix ill-conditioned
• Ill-conditioned matrix often yields excessive deformation
• Temporarily reduce the penalty term in the stiffness calculation
• Stress calculation use the penalty term as it is
Example - Hydrostatic Tension (Dilatation)

\[
\begin{align*}
\begin{cases}
x_1 = \alpha x_1 \\
x_2 = \alpha x_2 \\
x_3 = \alpha x_3
\end{cases}
\end{align*}
\quad
\begin{align*}
F = \begin{bmatrix}
\alpha & 0 & 0 \\
0 & \alpha & 0 \\
0 & 0 & \alpha
\end{bmatrix}
\quad
C = \begin{bmatrix}
\alpha^2 & 0 & 0 \\
0 & \alpha^2 & 0 \\
0 & 0 & \alpha^2
\end{bmatrix}
\end{align*}
\]

• Invariants

\begin{align*}
I_1 &= 3\alpha^2 \quad I_2 = 3\alpha^4 \quad I_3 = \alpha^6 \\
I_1 \text{ and } I_2 \text{ are not constant}
\end{align*}

• Reduced invariants

\begin{align*}
J_1 &= I_1 I_3^{-1/3} = 3 \quad J_1 \text{ and } J_2 \text{ are constant} \\
J_2 &= I_2 I_3^{-2/3} = 3 \\
J_3 &= I_3^{1/2} = \alpha^3
\end{align*}

Strain Energy Density

• Using reduced invariants

\[
W(J_1, J_2, J_3) = W_d(J_1, J_2) + W_H(J_3)
\]

- \(W_d(J_1, J_2)\): Distortional strain energy density
- \(W_H(J_3)\): Dilatational strain energy density

• The second terms is related to nearly incompressible behavior

\[
W_H(J_3) = \frac{K}{2} (J_3 - 1)^2
\]

- \(K\): bulk modulus = \(\lambda + \frac{2}{3} \mu\) for linear elastic material

Abaqus: \[W_H(J_3) = \frac{1}{2D} (J_3 - 1)^2\]
Mooney-Rivlin Material

• Most popular model
  - (not because accuracy, but because convenience)
    \[ W(J_1, J_2, J_3) = W_D(J_1, J_2) + W_H(J_3) \]
    \[ = A_{10}(J_1 - 3) + A_{01}(J_2 - 3) + \frac{K}{2}(J_3 - 1)^2 \]
  - Initial shear modulus \( \sim 2(A_{10} + A_{01}) \)
  - Initial Young's modulus \( \sim 6(A_{10} + A_{01}) \) (3D) or \( 8(A_{10} + A_{01}) \) (2D)
  - Bulk modulus = \( K \)

• Hydrostatic pressure
  \[ p = \frac{\partial W}{\partial J_3} = \frac{\partial W_H}{\partial J_3} = K(J_3 - 1) \]
  - Numerical instability for large \( K \) (volumetric locking)
  - Penalty method with \( K \) as a penalty parameter

Mooney-Rivlin Material cont.

• Second P-K stress
  \[ \mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} = \frac{\partial W}{\partial J_1} \frac{\partial J_1}{\partial \mathbf{E}} + \frac{\partial W}{\partial J_2} \frac{\partial J_2}{\partial \mathbf{E}} + \frac{\partial W}{\partial J_3} \frac{\partial J_3}{\partial \mathbf{E}} \]
  \[ \mathbf{S} = A_{10}J_{1,E} + A_{01}J_{2,E} + K(J_3 - 1)J_{3,E} \]
  - Use chain rule of differentiation
    \[ J_{1,E} = (I_3^{-1/3})I_{1,E} - \frac{1}{3} I_1(I_3^{-4/3})I_{3,E} \]
    \[ J_{2,E} = (I_3^{-2/3})I_{2,E} - \frac{2}{3} I_2(I_3^{-5/3})I_{3,E} \]
    \[ J_{3,E} = \frac{1}{2}(I_3^{-1/2})I_{3,E} \]

  \[ I_{1,E} = 21 \]
  \[ I_{2,E} = 4(1 + tr\mathbf{E})1 - 4\mathbf{E} \]
  \[ I_{3,E} = (2 + 4tr\mathbf{E})1 - 4\mathbf{E} + \left[ \frac{9}{4} e_{imn} e_{jrs} E_m r E_{ns} \right] \]

  \[ J_1 = I_1 I_3^{-1/3} \]
  \[ J_2 = I_2 I_3^{-2/3} \]
  \[ J_3 = I_3^{1/2} \]

  \[ I_{1,E} = 21 \]
  \[ I_{2,E} = 2(I_1 1 - C) \]
  \[ I_{3,E} = 2I_3 C^{-1} \]
Example

• Show $I_{1,E} = 21$, $I_{2,E} = 2(I_{1} - C)$, $I_{3,E} = 2I_{3}C^{-1}$

• Let $\bar{I}_1 = \text{tr}(C)$, $\bar{I}_2 = \frac{1}{2} \text{tr}(CC)$, $\bar{I}_3 = \frac{1}{3} \text{tr}(CCC)$

• Then $I_1 = \bar{I}_1$, $I_2 = \frac{1}{2} \bar{I}_1^2 - \bar{I}_2$, $I_3 = \bar{I}_3 + \frac{1}{6} \bar{I}_1^3 - \bar{I}_1 \bar{I}_2$

• Derivatives

\[
\frac{\partial \bar{I}_1}{\partial C_{ij}} = \delta_{ij}, \quad \frac{\partial \bar{I}_2}{\partial C_{ij}} = C_{ji}, \quad \frac{\partial \bar{I}_3}{\partial C_{ij}} = C_{jk}C_{ki}
\]

\[
\frac{\partial I_1}{\partial C_{ij}} = \delta_{ij}, \quad \frac{\partial I_2}{\partial C_{ij}} = I_1 \delta_{ij} - C_{ji}, \quad \frac{\partial I_3}{\partial C_{ij}} = I_3 C_{ji}^{-1}
\]

and

\[
\frac{\partial}{\partial C} = 2 \frac{\partial}{\partial E}
\]

Mixed Formulation

• Using bulk modulus often causes instability
  - Selectively reduced integration (Full integration for deviatoric part, reduced integration for dilatation part)

• Mixed formulation: Independent treatment of pressure

\[
W_{H}(J_3, p) = p(J_3 - 1)
\]

  - Pressure $p$ is additional unknown (pure incompressible material)
  - Advantage: No numerical instability
  - Disadvantage: system matrix is not positive definite

• Perturbed Lagrangian formulation

\[
W_{H}(J_3, p) = p(J_3 - 1) - \frac{1}{2K}p^2
\]

  - Second term make the material nearly incompressible and the system matrix positive definite
Variational Equation (Perturbed Lagrangian)

- Stress calculation

\[ W(J_1, J_2, J_3) = A_{10}(J_1 - 3) + A_{01}(J_2 - 3) + p(J_3 - 1) + \frac{1}{2K}p^2 \]

\[ S = A_{10}J_{1,E} + A_{01}J_{2,E} + pJ_{3,E} \]

- Variation of strain energy density

\[ \bar{W} = W_E \bar{E} + W_p \bar{p} \]

\[ = S : \bar{E} + (J_3 - 1 - \frac{p}{K})\bar{p} \]

- Introduce a vector of unknowns: \( r = (u, p) \)

\[ a(r, \bar{r}) = \int_\Omega \left[ S : \bar{E} + p\bar{H} \right] d\Omega \]

\[ H = J_3 - 1 - \frac{p}{K} \quad \text{Volumetric strain} \]

Example – Simple Shear

- Calculate 2\textsuperscript{nd} P-K stress for the simple shear deformation

- Material properties \( (A_{10}, A_{01}, K) \)

\[ F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ C = F^T F = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ I_1 = 4, \quad I_2 = 4, \quad I_3 = 1 \]

\[ I_{1,E} = 21 \]

\[ I_{2,E} = 2(I_1 - C) = \begin{bmatrix} 6 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \]

\[ I_{3,E} = 2I_3C^{-1} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \]
Example – Simple Shear cont.

\[ J_1 = I_1 I_3^{-1/3} = 4 \quad J_{1,E} = I_{1,E} - \frac{4}{3} I_{3,E} = \begin{bmatrix} -5 & 4 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \]

\[ J_2 = I_2 I_3^{2/3} = 4 \quad J_{2,E} = I_{2,E} - \frac{8}{3} I_{3,E} = \begin{bmatrix} -7 & 5 & 0 \\ 5 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ S = A_{10} J_{1,E} + A_{01} J_{2,E} + K (J_3 - 1) J_{3,E} \]

\[ = \frac{2}{3} \begin{bmatrix} -5A_{10} - 7A_{01} & 4A_{10} + 5A_{01} & 0 \\ 4A_{10} + 5A_{01} & -A_{10} - 2A_{01} & 0 \\ 0 & 0 & -A_{10} + A_{01} \end{bmatrix} \]

Note: \( S_{11}, S_{22} \) and \( S_{33} \) are not zero

Stress Calculation Algorithm

- Given: \( \{E\} = \{E_{11}, E_{22}, E_{33}, E_{12}, E_{23}, E_{13}\}^T, \{p\}, (A_{10}, A_{01}) \)

\[
\{1\} = \{1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0\}^T \quad \{C\} = 2\{E\} + \{1\}
\]

\[
I_1 = C_1 + C_2 + C_3 \\
I_2 = C_1 C_2 + C_1 C_3 + C_2 C_3 - C_4 C_4 - C_5 C_5 - C_6 C_6 \\
I_3 = (C_1 C_2 - C_4 C_4) C_3 + (C_4 C_6 - C_1 C_5) C_5 + (C_4 C_5 - C_2 C_6) C_6
\]

\[
\{I_{1,E}\} = 2\{1 \quad 1 \quad 1 \quad 0\} \\
\{I_{2,E}\} = 2\{C_2 + C_3 \quad C_3 + C_1 \quad C_1 + C_2 \quad -C_4 \quad -C_5 \quad -C_6\} \\
\{I_{3,E}\} = 2\{C_2 C_3 - C_5^2 \quad C_3 C_1 - C_6^2 \quad C_1 C_2 - C_4^2 \quad C_5 C_6 - C_3 C_4 \quad C_6 C_4 - C_1 C_5 \quad C_4 C_5 - C_2 C_6\}
\]

\[
\{J_{1,E}\} = I_3^{-1/3}\{I_{1,E}\} - \frac{1}{3} I_4 I_3^{-4/3}\{I_{3,E}\} \\
\{J_{2,E}\} = I_3^{-2/3}\{I_{2,E}\} - \frac{2}{3} I_5 I_3^{-5/3}\{I_{3,E}\} \\
\{J_{3,E}\} = \frac{1}{2} I_3^{-1/2}\{I_{3,E}\},
\]

For penalty method, use \( K(J_3 - 1) \) instead of \( p \)
**Linearization (Penalty Method)**

- **Stress increment**
  \[ \Delta S = W_{E,E} : \Delta E = D : \Delta E \]

- **Material stiffness**
  \[ D = \frac{\partial S}{\partial E} = A_0 J_{1,EE} + A_{01} J_{2,EE} + K(J_3 - 1) J_{3,EE} + K J_{3,E} \otimes J_{3,E} \]

- **Linearized energy form**
  \[ a^*(\dot{u}; \Delta u, \bar{u}) = \int_{\Omega_0} \left[ \bar{E} : D : \Delta E + S : \Delta \bar{E} \right] d\Omega \]

**Linearization cont.**

- **Second-order derivatives of reduced invariants**
  
  \[ J_{1,EE} = I_{1,EE} I_3^{\frac{1}{3}} - \frac{1}{3} I_3^{\frac{4}{3}} (I_{1,E} \otimes I_{3,E} + I_{3,E} \otimes I_{1,E}) + \frac{4}{9} I_1 I_3^{\frac{7}{3}} I_{3,E} \otimes I_{3,E} = \frac{1}{3} I_1 I_3^{\frac{7}{3}} I_{3,EE} \]
  
  \[ J_{2,EE} = I_{2,EE} I_3^{\frac{2}{3}} - \frac{2}{3} I_3^{\frac{5}{3}} (I_{2,E} \otimes I_{3,E} + I_{3,E} \otimes I_{2,E}) + \frac{10}{9} I_2 I_3^{\frac{8}{3}} I_{3,E} \otimes I_{3,E} = \frac{2}{3} I_2 I_3^{\frac{8}{3}} I_{3,EE} \]
  
  \[ J_{3,EE} = -\frac{1}{4} I_3^{\frac{3}{2}} I_{3,E} \otimes I_{3,E} + \frac{1}{2} I_3^{\frac{1}{2}} I_{3,EE} \]

\[ I_{1,EE} = 0 \]

\[ I_{2,EE} = 4I \otimes 1 - I \]

\[ I_{3,EE} = 4I_3^{-1} \otimes C^{-1} - I_3 C^{-1} IC^{-1} \]
MATLAB Function Mooney

- **Calculates S and D for a given deformation gradient**

```matlab
% 2nd PK stress and material stiffness for Mooney-Rivlin material

function [Stress D] = Mooney(F, A10, A01, K, ltan)
% Inputs:
%  F = Deformation gradient [3x3]
%  A10, A01, K = Material constants
%  ltan = 0 Calculate stress alone;
%         1 Calculate stress and material stiffness
% Outputs:
%  Stress = 2nd PK stress [S11, S22, S33, S12, S23, S13];
%  D = Material stiffness [6x6]
```

Summary

- **Hyperelastic material**: strain energy density exists with incompressible constraint
- In order to be material frame indifferent, material properties must be expressed using invariants
- Numerical instability (volumetric locking) can occur when large bulk modulus is used for incompressibility
- **Mixed formulation** is used for purely incompressibility (additional pressure variable, non-PD tangent stiffness)
- Perturbed Lagrangian formulation for nearly incompressibility (reduced integration for pressure term)
Section 3.6
Finite Element Formulation for Nonlinear Elasticity

Voigt Notation

- We will use the Voigt notation because the tensor notation is not convenient for implementation
  - 2nd-order tensor ⇒ vector
  - 4th-order tensor ⇒ matrix

- Stress and strain vectors (Voigt notation)

\[
\{S\} = \{S_{11} \quad S_{22} \quad S_{12}\}^T \\
\{E\} = \{E_{11} \quad E_{22} \quad 2E_{12}\}^T
\]

- Since stress and strain are symmetric, we don't need 21 component
4-Node Quadrilateral Element in TL

- We will use plane-strain, 4-node quadrilateral element to discuss implementation of nonlinear elastic FEA
- We will use TL formulation
- UL formulation will be discussed in Chapter 4

Interpolation and Isoparametric Mapping

- Displacement interpolation
  \[ u = \sum_{I=1}^{N_e} N_I(s)u_I \]
  Nodal displacement vector \((u_I, v_I)\)

- Isoparametric mapping
  - The same interpolation function is used for geometry mapping
  \[ X = \sum_{I=1}^{N_e} N_I(s)X_I \]
  Nodal coordinate \((X_I, Y_I)\)

Interpolation (shape) function
- Same for all elements
- Mapping depends on geometry

\[ N_1 = \frac{1}{4}(1 - s)(1 - t) \]
\[ N_2 = \frac{1}{4}(1 + s)(1 - t) \]
\[ N_3 = \frac{1}{4}(1 + s)(1 + t) \]
\[ N_4 = \frac{1}{4}(1 - s)(1 + t) \]
Displacement and Deformation Gradients

- **Displacement gradient**

\[
\frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \sum_{I=1}^{N_s} \frac{\partial N_I(s)}{\partial x} \mathbf{u}_I, \quad u_{i,j} = \sum_{I=1}^{N_s} N_{I,j}(s) u_{I,i}.
\]

\[\nabla_0 \mathbf{u} = \{u_{1,1}, u_{1,2}, u_{2,1}, u_{2,2}\}^T\]

- How to calculate \(\frac{\partial N_I(s)}{\partial x}\)?

- **Deformation gradient**

\[
\{\mathbf{F}\} = \{F_{11}, F_{12}, F_{21}, F_{22}\}^T = \{1 + u_{1,1}, u_{1,2}, u_{2,1}, 1 + u_{2,2}\}^T.
\]

- Both displacement and deformation gradients are not symmetric.

Green-Lagrange Strain

- **Green-Lagrange strain**

\[\{\mathbf{E}\} = \begin{bmatrix} E_{11} \\ E_{22} \\ 2E_{12} \end{bmatrix} = \begin{bmatrix} u_{1,1} + \frac{1}{2}(u_{1,1} u_{1,1} + u_{2,1} u_{2,1}) \\ u_{2,2} + \frac{1}{2}(u_{1,2} u_{2,1} + u_{2,2} u_{2,2}) \\ u_{1,2} + u_{2,1} + u_{1,2} u_{1,1} + u_{2,1} u_{2,2} \end{bmatrix}\]

- Due to nonlinearity, \(\{\mathbf{E}\} \neq \mathbf{[B]}\{\mathbf{d}\}\)

- For St. Venant-Kirchhoff material, \(\{\mathbf{S}\} = \mathbf{[D]}\{\mathbf{E}\}\)

\[
[D] = \begin{bmatrix}
\lambda + 2\mu & \lambda & 0 \\
\lambda & \lambda + 2\mu & 0 \\
0 & 0 & \mu
\end{bmatrix}
\]
Variation of G-R Strain

- Although $E(u)$ is nonlinear, $\bar{E}(u, \bar{u})$ is linear

$$\bar{E}(u, \bar{u}) = \text{sym}((\nabla_0 \bar{u})^T F)$$

$$\{\bar{E}\} = [B_N]\{\bar{d}\}$$

$$[B_N] = \begin{bmatrix}
F_{11}N_{1,1} & F_{21}N_{1,1} & F_{11}N_{2,1} & F_{21}N_{2,1} & \cdots & F_{11}N_{4,1} & F_{21}N_{4,1} \\
F_{12}N_{1,2} & F_{22}N_{1,2} & F_{12}N_{2,2} & F_{22}N_{2,2} & \cdots & F_{12}N_{4,2} & F_{22}N_{4,2} \\
F_{11}N_{1,2} & F_{21}N_{1,2} & F_{11}N_{2,2} & F_{21}N_{2,2} & \cdots & F_{11}N_{4,2} & F_{21}N_{4,2} \\
+ F_{12}N_{1,1} & F_{22}N_{1,1} & F_{12}N_{2,1} & F_{22}N_{2,1} & \cdots & F_{12}N_{4,1} & F_{22}N_{4,1}
\end{bmatrix}$$

Function of $u$
Different from linear strain-displacement matrix

Variational Equation

- Energy form

$$a(u, \bar{u}) = \iint_{\Omega_0} S : \bar{E} \, d\Omega$$

$$\approx \{\bar{d}\}^T \iint_{\Omega_0} [B_N]^T \{S\} \, d\Omega$$

$$\equiv \{\bar{d}\}^T \{F^{\text{int}}\}$$

- Load form

$$\ell(\bar{u}) = \iint_{\Omega_0} \bar{u}^T f^b \, d\Omega + \int_{\Gamma_0} \bar{u}^T \bar{t} \, d\Gamma$$

$$\approx \sum_{I=1}^{N_0} \bar{u}_I^T \left\{ \int_{\Omega_0} N_I(s) f^b \, d\Omega + \int_{\Gamma_0} N_I(s) \bar{t} \, d\Gamma \right\}$$

$$\equiv \{\bar{d}\}^T \{F^{\text{ext}}\}$$

- Residual

$$\{\bar{d}\}^T \{F^{\text{int}}(\bar{d})\} = \{\bar{d}\}^T \{F^{\text{ext}}\}, \quad \forall \{\bar{d}\} \in \mathbb{Z}_h$$
Linearization – Tangent Stiffness

- Incremental strain \( \{\Delta E\} = [B_N]\{\Delta d\} \)
- Linearization

\[
\iint_{\Omega_0} \overline{E} : D : \Delta E \, d\Omega = \{\overline{d}\}^T \left[ \iint_{\Omega_0} [B_N]^T [D][B_N] \, d\Omega \right] \{\Delta d\}
\]

\[
\iint_{\Omega_0} S : \Delta \overline{E} \, d\Omega = \{\overline{d}\}^T \left[ \iint_{\Omega_0} [B_G]^T [\Sigma][B_G] \, d\Omega \right] \{\Delta d\}
\]

\[
[S] = \begin{bmatrix}
S_{11} & S_{12} & 0 & 0 \\
S_{12} & S_{22} & 0 & 0 \\
0 & 0 & S_{11} & S_{12} \\
0 & 0 & S_{12} & S_{22}
\end{bmatrix}
\]

\[
[B_G] = \begin{bmatrix}
N_{1,1} & 0 & N_{2,1} & 0 & N_{3,1} & 0 & N_{4,1} & 0 \\
N_{1,2} & 0 & N_{2,2} & 0 & N_{3,2} & 0 & N_{4,2} & 0 \\
0 & N_{1,1} & 0 & N_{2,1} & 0 & N_{3,1} & 0 & N_{4,1} \\
0 & N_{2,1} & 0 & N_{2,2} & 0 & N_{3,2} & 0 & N_{4,2}
\end{bmatrix}
\]

\[
[K_T] = \iint_{\Omega_0} \left[ [B_N]^T [D][B_N] + [B_G]^T [\Sigma][B_G] \right] \, d\Omega_0
\]

- Tangent stiffness

- Discrete incremental equation (N-R iteration)

\[
\{\overline{d}\}^T [K_T] \{\Delta d\} = \{\overline{d}\}^T \{F^{ext} - F^{int}\}, \quad \forall \{\overline{d}\} \in Z_h
\]

- \([K_T]\) changes according to stress and strain
- Solved iteratively until the residual term vanishes
Summary

• For elastic material, the variational equation can be obtained from the principle of minimum potential energy.

• St. Venant-Kirchhoff material has linear relationship between 2\textsuperscript{nd} P-K stress and G-L strain.

• In TL, nonlinearity comes from nonlinear strain-displacement relation.

• In UL, nonlinearity comes from constitutive relation and unknown current domain (Jacobian of deformation gradient).

• TL and UL are mathematically equivalent, but have different reference frames.

• TL and UL have different interpretation of constitutive relation.

Section 3.7
MATLAB Code for Hyperelastic Material Model
• Building the tangent stiffness matrix, \( K \), and the residual force vector, \( \{R\} \), for hyperelastic material

• Input variables for HYPER3D.m

<table>
<thead>
<tr>
<th>Variable</th>
<th>Array size</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>MID</td>
<td>Integer</td>
<td>Material Identification No. (3) (Not used)</td>
</tr>
<tr>
<td>PROP</td>
<td>(3,1)</td>
<td>Material properties ((A_{10}, A_{01}, K))</td>
</tr>
<tr>
<td>UPDATE</td>
<td>Logical variable</td>
<td>If true, save stress values</td>
</tr>
<tr>
<td>LTAN</td>
<td>Logical variable</td>
<td>If true, calculate the global stiffness matrix</td>
</tr>
<tr>
<td>NE</td>
<td>Integer</td>
<td>Total number of elements</td>
</tr>
<tr>
<td>NDOF</td>
<td>Integer</td>
<td>Dimension of problem (3)</td>
</tr>
<tr>
<td>XYZ</td>
<td>(3,NNODE)</td>
<td>Coordinates of all nodes</td>
</tr>
<tr>
<td>LE</td>
<td>(8,NE)</td>
<td>Element connectivity</td>
</tr>
</tbody>
</table>

function HYPER3D(MID, PROP, UPDATE, LTAN, NE, NDOF, XYZ, LE)

%***********************************************************************
% MAIN PROGRAM COMPUTING GLOBAL STIFFNESS MATRIX AND RESIDUAL FORCE FOR
% HYPERELASTIC MATERIAL MODELS
%***********************************************************************

global DISPTD FORCE GKF SIGMA

% Integration points and weights
XG=[-0.57735026918963D0, 0.57735026918963D0];
WGT=[1.00000000000000D0, 1.00000000000000D0];

% Index for history variables (each integration pt)
INTN=0;

% LOOP OVER ELEMENTS, THIS IS MAIN LOOP TO COMPUTE \( K \) AND \( F \)
for IE=1:NE
    % Nodal coordinates and incremental displacements
    ELXY=XY2(LE(IE,:,:),:); % Local to global mapping
    IDOF=zeros(1,24);
    for I=1:8
        II=(I-1)*NDOF+1;
        IDOF(II:II+2)=(LE(IE,I)-1)*NDOF+1:(LE(IE,I)-1)*NDOF+3;
    end
    DSP=DISPTD(IDOF);
    DSP=reshape(DSP,NDOF,8);

    % LOOP OVER INTEGRATION POINTS
    for LX=1:2, for LY=1:2, for LZ=1:2
        E1=XG(LX); E2=XG(LY); E3=XG(LZ);
        INTN = INTN + 1;
        % Determinant and shape function derivatives
        [~, SHPD, DET] = SHAPEL([E1 E2 E3], ELXY);
        FAC=WGT(LX)*WGT(LY)*WGT(LZ)*DET;
        ...
% Deformation gradient
F = DSP * SHPD * + eye(3);
%
% Computer stress and tangent stiffness
[STRESS DTAN] = MooneyF, PROP(1), PROP(2), PROP(3), LTAN);
%
% Store stress into the global array
if UPDATE
SIGMA(:, INTN) = STRESS;
continue;
end
%
% Add residual force and tangent stiffness matrix
BM = zeros(6, 24); BG = zeros(9, 24);
for I = 1:8
COL = (I-1) * 3 + 1: (I-1) * 3 + 3;
BM(:, COL) = [SHPD(1, I) * F(1, 1) SHPD(1, I) * F(2, 1) SHPD(1, I) * F(3, 1)];
SHPD(2, I) * F(1, 2) SHPD(2, I) * F(2, 2) SHPD(2, I) * F(3, 2);
SHPD(3, I) * F(1, 3) SHPD(3, I) * F(2, 3) SHPD(3, I) * F(3, 3);
SHPD(1, I) * F(1, 2) + SHPD(2, I) * F(1, 1)
SHPD(1, I) * F(2, 2) + SHPD(2, I) * F(2, 1) SHPD(1, I) * F(3, 2) + SHPD(2, I) * F(3, 1);
SHPD(1, I) * F(1, 3) + SHPD(2, I) * F(1, 2)
SHPD(2, I) * F(2, 3) + SHPD(3, I) * F(2, 2) SHPD(2, I) * F(3, 3) + SHPD(3, I) * F(3, 2);
SHPD(1, I) * F(1, 3) + SHPD(2, I) * F(1, 2)
SHPD(2, I) * F(2, 3) + SHPD(3, I) * F(2, 2)
SHPD(3, I) * F(3, 3) + SHPD(3, I) * F(3, 1)];
%
BG(:, COL) = [SHPD(1, I) 0 0;
SHPD(2, I) 0 0;
SHPD(3, I) 0 0;
0 SHPD(1, I) 0;
0 SHPD(2, I) 0;
0 SHPD(3, I) 0;
0 0 SHPD(1, I);
0 0 SHPD(2, I);
0 0 SHPD(3, I)];
end
%
% Residual forces
FORCE(IDOF) = FORCE(IDOF) - FAC * BM' * STRESS;
%
% Tangent stiffness
if LTAN
SIG = [STRESS(1) STRESS(4) STRESS(6);
STRESS(4) STRESS(2) STRESS(5);
STRESS(6) STRESS(5) STRESS(3)];
SHEAD = zeros(9);
SHEAD(1:3, 1:3) = SIG;
SHEAD(4:6, 4:6) = SIG;
SHEAD(7:9, 7:9) = SIG;
%
EKF = BM' * DTAN * BM + BG' * SHEAD * BG;
GRF (IDOF, IDOF) = GRF (IDOF, IDOF) + FAC * EKF;
end
end; end; end; end
Example Extension of a Unit Cube

- Face 4 is extended with a stretch ratio $\lambda = 6.0$
- BC: $u_1 = 0$ at Face 6, $u_2 = 0$ at Face 3, and $u_3 = 0$ at Face 1
- Mooney-Rivlin: $A_{10} = 80\text{MPa}$, $A_{01} = 20\text{MPa}$, and $K = 10^7$

% Nodal coordinates
XYZ=[0 0 0;1 0 0;1 1 0;0 1 0;0 0 1;1 1 1;0 1 1];

% Element connectivity
LE=[1 2 3 4 5 6 7 8];

% No external force
EXTFORCE=[];

% Prescribed displacements [Node, DOF, Value]
SDISPT=[1 1 0;4 1 0;5 1 0;8 1 0; % $u_1=0$ for Face 6
1 2 0;2 2 0;5 2 0;6 2 0; % $u_2=0$ for Face 3
1 3 0;2 3 0;3 3 0;4 3 0; % $u_3=0$ for Face 1
2 1 5;3 1 5;6 1 5;7 1 5]; % $u_1=5$ for Face 4

% Load increments [Start End Increment InitialFactor FinalFactor]
TIMS=[0.0 1.0 0.05 0.0 1.0]';

% Material properties
MID=-1;
PROP=[80 20 1E7];

---

Example Extension of a Unit Cube

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<thead>
<tr>
<th>Time</th>
<th>Time step</th>
<th>Iter</th>
<th>Residual</th>
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<tbody>
<tr>
<td>0.05000</td>
<td>5.000e-02</td>
<td>2</td>
<td>1.17493e+05</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Not converged. Bisecting load increment 2</td>
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<table>
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<th>Time step</th>
<th>Iter</th>
<th>Residual</th>
</tr>
</thead>
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<td>2</td>
<td>2.96114e+04</td>
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<tr>
<td></td>
<td></td>
<td>3</td>
<td>2.55611e+02</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>1.84747e-02</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>1.51867e-10</td>
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</table>

<table>
<thead>
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<th>Time</th>
<th>Time step</th>
<th>Iter</th>
<th>Residual</th>
</tr>
</thead>
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<td>2</td>
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<td></td>
<td></td>
<td>3</td>
<td>1.69171e+02</td>
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<td></td>
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<td>4</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>2.39898e-10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Time step</th>
<th>Iter</th>
<th>Residual</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>1.86783e-07</td>
</tr>
</tbody>
</table>

...
Hyperelastic Material Analysis Using ABAQUS

- **ELEMENT, TYPE=C3D8RH, ELSET=ONE**
  - 8-node linear brick, reduced integration with hourglass control, hybrid with constant pressure

- **MATERIAL, NAME=MOONEY**
  *HYPERELASTIC, MOONEY-RIVLIN 80., 20.1*
  - Mooney-Rivlin material with $A_{10} = 80$ and $A_{01} = 20$

- **STATIC, DIRECT**
  - Fixed time step (no automatic time step control)

---

Hyperelastic Material Analysis Using ABAQUS

*HEADING*
- Incompressible hyperelasticity (Mooney-Rivlin) Uniaxial tension

*NODE, NSET=ALL*
1, 2, 3, 4, 5, 6, 7, 8

*NSET, NSET=FACE1*
1, 2, 3, 4

*NSET, NSET=FACE3*
5, 6, 7, 8

*NSET, NSET=FACE4*
1, 2, 3, 4, 5

*NSET, NSET=FACE6*
1, 2, 3

*ELEMENT, TYPE=C3D8RH, ELSET=ONE*
1, 2, 3, 4, 5, 6, 7, 8

*SOLID SECTION, ELSET=ONE, MATERIAL= MOONEY*

*MATERIAL, NAME=MOONEY*

*HYPERELASTIC, MOONEY-RIVLIN 80., 20.1*

*STEP, NLGEOM, INC=20 UNIAXIAL TENSION*

*STATIC, DIRECT*
1, 2, 0, 20

*BOUNDARY, OP=NEW*
FACE1, 2
FACE2, 3
FACE6, 1
FACE4, 1, 1, 5

*EL PRINT, F=1*
S, E

*NODE PRINT, F=1*
U, RF

*OUTPUT, FIELD, FREQ=1*
S, E

*ELEMENT OUTPUT*

*OUTPUT, FIELD, FREQ=1*
U, RF

*NODE OUTPUT*

*END STEP*
Hyperelastic Material Analysis Using ABAQUS

• Analytical solution procedure
  - Gradually increase the principal stretch $\lambda$ from 1 to 6
  - Deformation gradient
    \[
    F = \begin{bmatrix}
    \lambda & 0 & 0 \\
    0 & 1/\sqrt{\lambda} & 0 \\
    0 & 0 & 1/\sqrt{\lambda}
    \end{bmatrix}
    \]
  - Calculate $J_{1,E}$ and $J_{2,E}$
  - Calculate 2nd P-K stress
    \[ S = A_{10}J_{1,E} + A_{01}J_{2,E} \]
  - Calculate Cauchy stress
    \[ \sigma = \frac{1}{J}F \cdot S \cdot F^T \]
  - Remove the hydrostatic component of stress
    \[ \sigma_{11} = \sigma_{11} - \sigma_{22} \]

Hyperelastic Material Analysis Using ABAQUS

• Comparison with analytical stress vs. numerical stress
Section 3.9
Fitting Hyperelastic Material Parameters from Test Data

Elastomer Test Procedures

- Elastomer tests
  - simple tension, simple compression, equi-biaxial tension, simple shear, pure shear, and volumetric compression
Elastomer Tests

- Data type: Nominal stress vs. principal stretch

![Diagram of simple tension test](image1)

![Diagram of pure shear test](image2)

![Diagram of equal biaxial test](image3)

![Diagram of volumetric compression test](image4)

Data Preparation

- Need enough number of independent experimental data
  - No rank deficiency for curve fitting algorithm
- All tests measure principal stress and principle stretch

<table>
<thead>
<tr>
<th>Experiment Type</th>
<th>Stretch</th>
<th>Stress</th>
</tr>
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<tbody>
<tr>
<td>Uniaxial tension</td>
<td>Stretch ratio $\lambda = L/L_0$</td>
<td>Nominal stress $T^E = F/A_0$</td>
</tr>
<tr>
<td>Equi-biaxial tension</td>
<td>Stretch ratio $\lambda = L/L_0$ in y-direction</td>
<td>Nominal stress $T^E = F/A_0$ in y-direction</td>
</tr>
<tr>
<td>Pure shear test</td>
<td>Stretch ratio $\lambda = L/L_0$</td>
<td>Nominal stress $T^E = F/A_0$</td>
</tr>
<tr>
<td>Volumetric test</td>
<td>Compression ratio $\lambda = L/L_0$</td>
<td>Pressure $T^E = F/A_0$</td>
</tr>
</tbody>
</table>
Data Preparation cont.

• Uni-axial test \( \lambda_1 = \lambda, \quad \lambda_2 = \lambda_3 = 1 / \sqrt{\lambda} \)

\[
T = \frac{\partial U}{\partial \lambda} = 2(1 - \lambda^{-3})(A_{10}\lambda + A_{01})
\]

\[
T(A_{10}, A_{01}, \lambda) = \{x\}^T\{b\} = \begin{bmatrix} 2(\lambda - \lambda^{-2}) & 2(1 - \lambda^{-3}) \end{bmatrix} \begin{bmatrix} A_{10} \\ A_{01} \end{bmatrix}
\]

• Equi-biaxial test \( \lambda_1 = \lambda_2 = \lambda, \quad \lambda_3 = 1 / \lambda^2 \)

\[
T = \frac{1}{2} \frac{\partial U}{\partial \lambda} = 2(\lambda - \lambda^{-5})(A_{10} + \lambda^2 A_{01})
\]

• Pure shear test \( \lambda_1 = \lambda, \quad \lambda_2 = 1, \quad \lambda_3 = 1 / \lambda \)

\[
T = \frac{\partial U}{\partial \lambda} = 2(\lambda - \lambda^{-3})(A_{10} + A_{01})
\]

Data Preparation cont.

• Data Preparation

\[
\begin{array}{cccccccc}
\text{Type} & 1 & 1 & 1 & \ldots & 4 & 4 & \ldots & 4 \\
\lambda & \lambda_1 & \lambda_2 & \lambda_3 & \ldots & \lambda_i & \lambda_{i+1} & \ldots & \lambda_{\text{NDT}} \\
T^E & T_1^E & T_2^E & T_3^E & \ldots & T_i^E & T_{i+1}^E & \ldots & T_{\text{NDT}}^E
\end{array}
\]

• For Mooney-Rivlin material model, nominal stress is a linear function of material parameters \((A_{10}, A_{01})\)
Curve Fitting for Mooney-Rivlin Material

- Need to determine $A_{10}$ and $A_{01}$ by minimizing error between test data and model

$$\text{minimize} \ A_{10}, A_{01} \ \sum_{k=1}^{NDT} \left( T_k^E - T(A_{10}, A_{01}, \lambda_k) \right)^2$$

- For Mooney-Rivlin, $T(A_{10}, A_{01}, \lambda_k)$ is linear function
  - Least-squares can be used

$$\{b\} = \begin{bmatrix} A_{10} \\ A_{01} \end{bmatrix}$$

$$\{T\} = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{NDT} \end{bmatrix} = \begin{bmatrix} x(\lambda_1)^T \\ x(\lambda_1)^T \\ \vdots \\ x(\lambda_{NDT})^T \end{bmatrix}$$

$$\{T\} = \{T\}^T \{T\} - 2\{b\}^T[X]^T\{T\} + \{b\}^T[X]^T[X]\{b\}$$

Curve Fitting cont.

- Minimize error(square)

$$\{e\}^T \{e\} = \{T^E - T\}^T \{T^E - T\}$$

$$= \{T^E - Xb\}^T \{T^E - Xb\}$$

$$= \{T^E\}^T \{T^E\} - 2\{b\}^T[X]^T\{T^E\} + \{b\}^T[X]^T[X]\{b\}$$

- Minimization $\rightarrow$ Linear regression equation

$$[X]^T[X]\{b\} = [X]^T\{T^E\}$$
Stability of Constitutive Model

- Stable material: the slope in the stress-strain curve is always positive (Drucker stability)

- Stability requirement (Mooney-Rivlin material)

\[ \varepsilon : D : \varepsilon > 0 \]

- Stability check is normally performed at several specified deformations (principal directions)

\[ d\sigma_1 d\varepsilon_1 + d\sigma_2 d\varepsilon_2 > 0 \]

\[
\begin{bmatrix}
   \varepsilon_1 \\
   \varepsilon_2
\end{bmatrix}
\begin{bmatrix}
   D_{11} & D_{12} \\
   D_{21} & D_{22}
\end{bmatrix}
\begin{bmatrix}
   \varepsilon_1 \\
   \varepsilon_2
\end{bmatrix} > 0
\]

- In order to be P.D.

\[ D_{11} + D_{22} > 0 \]

\[ D_{11} D_{22} - D_{12} D_{21} > 0 \]