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Scaling of Low-Velocity Impact for Symmetric Composite Laminates*

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ABSTRACT: The equations governing the problem of low-velocity impact of a simply supported rectangular midplane-symmetric laminated plate are nondimensionalized such that the problem is defined in terms of five dimensionless parameters. A parametric study using the Graeco-Latin Factorial Plan is performed. Semi-empirical formulas for maximum impact force, impact duration, and maximum back surface strains are obtained. It is found that some of the simple impact models provide the bounds for the case of impact on a finite extent plate. A one parameter model is derived for impacts of short duration.

KEY WORDS: scaling of impact, low-velocity impact, composite laminates, nondimensional models.

1. INTRODUCTION

PAST STUDIES, BOTH experimental and numerical, have shown that the response of composite laminates to low-velocity projectile impact can be predicted accurately up to the point of initiation of damage. Estimation of total damage requires progressive damage models beyond the point of damage initiation. While research is progressing in the areas of prediction of post-impact strength as well as developing impact resistant materials, there is a need to develop simple design procedures for engineers involved in designing composite structures that are susceptible to foreign object impact damage. There are a number of parameters in the impact problem that govern the response and damage initiation. In order to understand the effect of each parameter, it will be convenient to identify fewer nondimensional groups of variables, and study the problem by varying the nondimensional parameters. Further, scaling of response from laboratory coupons to prototype structures will also be possible, if nondimensional models are used.

The earliest nondimensional model for low-velocity impact response was proposed by Zener [1] in 1941. Sun and Chattopadhyay [2] studied the effect of initial

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stresses on impact using dimensionless variables. References [3–8] are some examples of recently proposed nondimensional models for the impact problem. In what follows, a method for scaling of impact variables is suggested, so that the impact equations can be expressed in terms of five nondimensional parameters. Two of them describe the plate stiffness, two of them describe the impactor mass and velocity, and the fifth parameter defines the interaction between the impactor and the plate.

A new numerical algorithm is developed to solve the impact equations. A parametric study is performed by choosing five values for each of the five non-dimensional parameters. The Graeco-Latin Factorial Plan is used to obtain empirical formulas for maximum impact force, impact duration and maximum back surface strains. The formulas are compared with the simple spring-mass model and also with the case of impact on an half-space. Following Zener [1] and Olsson [7], a one parameter model is derived, which will be useful in the case of impacts of very short duration typical of very small projectiles.

2. IMPACT EQUATIONS

We will consider the problem of central impact on a simply supported rectangular symmetric laminated plate. The transverse deflection w at the center of the plate is given by:

$$w(t) = (4/m_p)\Sigma_{n=1,3,\ldots,}\ \Sigma_{m=1,3,\ldots,}\ (1/\omega_{mn}) \int_0^t f(t^*) \sin \omega_{mn}(t - t^*)dt^* \tag{1}$$

where m_p is the mass and ω_{mn} are the natural frequencies of the plate, t is the time variable, and $f(t)$ is the unknown impact force history. The equation of motion of the impactor is:

$$m\ d^2x/dt^2 = -f(t) \tag{2}$$

where m is the impactor mass and $x(t)$ is the impactor displacement. The interaction between the plate and the impactor at the contact region is defined by the elastostatic contact law given by:

$$f = \mathrm{kq}^\alpha \tag{3}$$

where q, the indentation, is given by $q = x - w$. For a homogeneous transversely isotropic plate, the exponent α in Equation (3) can be taken as 1.5 and the contact stiffness k can be approximated as:

$$k = 1.33E_z\sqrt{R} \tag{4}$$

where E_z is the Young's modulus of the plate in the thickness direction, and R is the radius of curvature of the rigid impactor. For a general laminate the contact stiffness k and the exponent α have to be found from indentation tests [9] or numerical simulation of indentation tests [10]. For a quasi-isotropic layup, e.g., $[0,+45,-45,90]_{ns}$, $\alpha = 1.5$ seems to be a reasonable approximation.

The natural frequencies ω_{mn} of a simply supported rectangular plate depends on the type of laminated plate theory used to describe the plate motion. If we assume classical plate theory, then for a midplane-symmetric laminate:

$$\omega_{mn}^2 = (\pi^4/\varrho)[D_{11}(m/a)^4 + 2(D_{12} + 2D_{66})(mn/ab)^2 + D_{22}(n/b)^4] \quad (5)$$

where D_{ij} are the laminate flexural stiffness coefficients, ϱ is the mass density per unit area of the plate, and a and b are the length and width of the rectangular laminate. If the effects of shear deformation are included, then additional stiffness terms, such as A_{44}, A_{45} and A_{55}, will be introduced in Equation (5), which, in turn, may increase the number of non-dimensional parameters needed to define the impact problem. A numerical algorithm to solve Equations (1–3) can be found in Reference [11].

One of the modes of impact damage is the back surface fiber splitting and fiber breakage. The maximum back surface strain can be considered as a controlling factor for predicting back surface damage. The shear strain at the center of the plate is identically equal to zero due to symmetry. The normal strains ϵ_{xx} and ϵ_{yy} at the center of the plate on the back surface are:

$$\begin{aligned}\epsilon_{xx} &= -(h/2)d^2w/dx^2 @ (x = a/2, y = b/2) \\ &= (2h/m_p)\Sigma_{m=1,3,\ldots}\Sigma_{n=1,3,\ldots}(1/\omega_{mn})(m\pi/a)^2 \\ &\times \int_0^t f(t^*) \sin \omega_{mn}(t - t^*)dt^* \end{aligned} \quad (6)$$

$$\begin{aligned}\epsilon_{yy} &= -(h/2)d^2w/dy^2 @ (x = a/2, y = b/2) \\ &= (2h/m_p)\Sigma_{m=1,3,\ldots}\Sigma_{n=1,3,\ldots}(1/\omega_{mn})(n\pi/b)^2 \\ &\times \int_0^t f(t^*) \sin \omega_{mn}(t - t^*)dt^* \end{aligned} \quad (7)$$

The damage that occurs in the vicinity of the contact region, such as matrix cracking, delamination initiation, are due to local stresses caused by indentation. These stresses can be approximated by the corresponding values obtained from a static contact analysis. The maximum contact pressure, which is at the center of contact for small contact areas, can be considered as a measure of severity of stresses due to static indentation. For transversely isotropic materials, the relation between the contact force f, and the contact radius c is:

$$f = k_c c^3 \quad (8)$$

where k_c depends on k and R. The maximum contact pressure p_0 at the center is then given by:

$$p_0 = 1.5f/\pi c^2 \tag{9}$$

We shall use the term impact parameters to describe the constants in the above set of equations. The laminate parameters are: a, b, h, ϱ, D_{11}, D_{12}, D_{22} and D_{66}. The parameters that describe the impactor are: m, v_0 (impact velocity) and R. The contact stiffness k depends on both the impactor and the target material. Thus, there are 12 impact parameters that describe the impact problem. We shall use the term impact variables to describe the output quantities: f, t_{max} (impact duration), w, x, ϵ_{xx}, ϵ_{yy}, v_r (rebound velocity). The purpose of an impact analysis is to solve the above equations for a given set of impact parameters in order to predict the maximum values of the impact variables. In the following section a model is proposed to describe the impact problem in terms of five non-dimensional parameters.

3. NON-DIMENSIONAL EQUATIONS

The mass, length and time dimensions can be normalized with respect to plate mass m_p, plate length a, and the fundamental period of the laminate $(1/\omega_{mn})$ respectively. Then Equations (1–3) can be written in terms of two nondimensional plate parameters C and D, and three nondimensional impactor parameters M, K, and V. The definition of the nondimensional parameters are:

$$C = 2(D_{12} + 2D_{66})a^2/D_{11}b^2,\ D = D_{22}a^4/D_{11}b^4,\ M = m/m_p,$$
$$V = v_0/a\omega_{11},\ K = k\sqrt{a}/m_p\omega_{11}^2 \tag{10}$$

The plate parameters C and D are a measure of the anisotropy in the flexural behavior of the plate. For isotropic square plates C and D are equal to 2 and 1 respectively. The nondimensional impact variables are:

$$T = \omega_{11}t,\ X = x/a,\ W = w/a,\ \Omega_{mn} = \omega_{mn}/\omega_{11},\ F = f/m_pa\omega_{11}^2$$
$$E_{xx} = a\epsilon_{xx}/h,\ E_{yy} = b^2\epsilon_{yy}/ah,\ Q = q/a \tag{11}$$

The expression for nondimensional frequencies take the form:

$$\Omega_{mn}^2 = (m^4 + Cm^2n^2 + Dn^4)/(1 + C + D). \tag{12}$$

Equations (1–3) can be written as:

$$W(T) = \Sigma_{m=1,3,\ldots}\ \Sigma_{n=1,3,\ldots}\ (4/\Omega_{mn}) \int_0^T F(T^*) \sin \Omega_{mn}(T - T^*)dT^* \tag{13}$$

$$M\ddot{X} = -F(T) \tag{14}$$

$$F = KQ^{\alpha} \tag{15}$$

with the initial condition $X(0) = 0$, and $\dot{X}(0) = V$. In the nondimensional model a dot over a variable denotes differentiation with respect to the nondimensional time T.

The back surface strains are:

$$E_{xx} = 2\Sigma_{m=1,3,\ldots}\ \Sigma_{n=1,3,\ldots}\ (m^2\pi^2/\Omega_{mn})\int_0^T F(T^*)\ \sin\ \Omega_{mn}(T - T^*)dT^* \tag{16}$$

$$E_{yy} = 2\Sigma_{m=1,3,\ldots}\ \Sigma_{n=1,3,\ldots}\ (n^2\pi^2/\Omega_{mn})\int_0^T F(T^*)\ \sin\ \Omega_{mn}(T - T^*)dT^* \tag{17}$$

In order to solve Equations (13–15), and also to compute strains in Equations (16) and (17), one can use the numerical algorithm given in Reference [11]. In the present study a slightly modified procedure is used, and it is explained in the Appendix.

4. SEMI-EMPIRICAL SOLUTION

Equations (13–15) are nonlinear integral equations in variables F, X and W. Since a closed form solution is not possible, we seek a semi-empirical solution that will be useful to designers, by conducting a systematic parametric study. Equations (13–15) were numerically solved for various values of the nondimensional parameters in a given range. Then, important impact variables such as the maximum value of the impact force, impact duration, maximum strain etc., were computed for each case. A curve fitting procedure was used to obtain a simplified empirical relation between the impact parameters and the impact variables.

Standard regression analysis procedures will require a large number of test cases to obtain a functional relationship between the impact variables and the impact parameters. For example, if we decide to use five values for each parameter, then 5^5 impact problems have to be solved to find the effect of each parameter on the maximum impact force and other impact variables. In the present study the Graeco-Latin Factorial Plan [12], which requires only 5^2 combination of the parameters, is used to obtain semi-empirical formulas for the impact problem. Traditionally factorial plans are used to curve-fit experimental data so that maximum information can be obtained from a minimum number of tests. Factorial plans have also been used to study numerical simulations [13].

We assume that the maximum value of any impact variable, for example maximum impact force F_{max}, can be expressed in the form:

$$F_{max} = F_0\phi_1(M)\phi_2(V)\phi_3(C)\phi_4(D)\phi_5(K) \tag{18}$$

where F_0 is a constant and ϕ_i are semi-empirical functions to be determined from the parametric study.

While selecting the values of C, D, M, V and K for the parametric study, care has to be taken to ensure that the range of each parameter is small enough for accurate curve fitting, but at the same time, the range should be useful in actual design situations. The impact problem was divided into two broad categories: (i) large mass (e.g., 1–10 kg) impacting at lower velocities (e.g., 1–3 m/s); (ii) small mass (e.g., 0.01–0.1 kg) impacting at higher velocities (10–100 m/s). The former can be considered to simulate impact damage due to dropped hand tools, whereas the latter corresponds to small projectile impacts. Five values were selected for each nondimensional impact parameter. They are shown in Table 1. The combination of parameters used in each of the 25 numerical test cases are shown in Table 2 (Graeco-Latin Square). It may be noted that the range and the values of parameters chosen in the present study are not unique, but they have been used only for the purpose of demonstrating the usefulness of the present approach.

From the studies described above the following empirical formulas for F_{max}, T_{max}, $E_{xx(max)}$ were obtained for both large and small mass impact.

Large Mass Impact

$$F_{max} = 0.4028M^{0.56}V^{1.11}K^{0.234} \quad (19)$$

$$T_{max} = 9.05M^{0.438}V^{-0.105}K^{-0.248} \quad (20)$$

$$E_{xx(max)} = 0.221M^{0.498}V^{1.17}(C + 36)(D + 21)K^{0.276} \quad (21)$$

Small Mass Impact

$$F_{max} = 0.4701M^{0.60}V^{1.14}K^{0.215} \quad (22)$$

$$T_{max} = 4.83M^{0.416}\ V^{-0.243}K^{-0.261} \quad (23)$$

$$E_{xx(max)} = 0.151M^{0.543}V^{1.17}(C + 50)(D + 14)K^{0.249} \quad (24)$$

It may be noted from Equations (19–20) and (22–23) that parameters C and D are not explicitly present in the expressions for F_{max} and T_{max}. This is because of the particular choice of dimensionless parameters used in the present study. It should be mentioned here that the formulas given above are valid only in the particular range of impact parameters shown in Table 1.

5. COMPARISON WITH SIMPLE MODELS

One can have a better appreciation of the quasi-empirical formulas given in Equations (19–24), if one considers some simple impact models. In the case of large mass impact, the target plate can be approximated by a spring-mass model, and the indentation stiffness can be assumed to be much higher than the plate

Table 1. Range of nondimensional parameters used in parametric studies.

Parameter	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
M_i (small mass)	0.02	0.06	0.2	0.6	2.0
M_i (large mass)	1.0	3.0	10.0	30.0	100
V_i (small mass)	0.001	0.003	0.008	0.025	0.07
V_i (large mass)	0.0002	0.0004	0.0008	0.0017	0.0035
C_i	0.1	0.4	1.4	5.0	18.0
D_i	0.002	0.03	0.4	5.0	75.0
K_i	0.1	0.3	1.0	3.0	10.0

Table 2. Graeco-Latin Square.

	M_1	M_2	M_3	M_4	M_5
V_1	1 C_1 D_1 K_5	2 C_2 D_2 K_3	3 C_3 D_3 K_2	4 C_4 D_4 K_1	5 C_5 D_5 K_4
V_2	6 C_5 D_2 K_1	7 C_1 D_3 K_4	8 C_2 D_4 K_5	9 C_3 D_5 K_3	10 C_4 D_1 K_2
V_3	11 C_4 D_3 K_3	12 C_5 D_4 K_2	13 C_1 D_5 K_1	14 C_2 D_1 K_4	15 C_3 D_2 K_5
V_4	16 C_3 D_4 K_4	17 C_4 D_5 K_5	18 C_5 D_1 K_3	19 C_1 D_2 K_2	20 C_2 D_3 K_1
V_5	21 C_2 D_5 K_2	22 C_3 D_1 K_1	23 C_4 D_2 K_4	24 C_5 D_3 K_5	25 C_1 D_4 K_3

stiffness. In the nondimensional model, the plate stiffness and equivalent mass, each will be equal to 0.25, so that the natural frequency of the spring-mass system is equal to unity. The impactor mass is M, and the impact velocity is V. Immediately after impact, the impactor and the plate mass attain a velocity of $MV/(M + 0.25)$ in order to conserve momentum. Assuming $4M \gg 1$, the simple harmonic motion of the spring-mass system is described as:

$$X = W = V \sin \Omega T \tag{25}$$

where $\Omega = 1/(1 + 4M)$. The expressions for the maximum impact force experienced by the impactor and the impact duration are:

$$F_{max} = 0.5VM^{0.5} \tag{26}$$

$$T_{max} = 6.28M^{0.5} \tag{27}$$

The impact of a smaller mass can be approximated by the impact on a half-space. The results for impact of a spherical mass on a half-space can be found in Reference [14]. The results expressed in nondimensional form are:

$$F_{max} = 1.143M^{0.6}V^{1.2}K^{0.4} \tag{28}$$

$$T_{max} = 3.21M^{0.4}V^{-0.2}K^{-0.4} \tag{29}$$

From these results it may be noted that the exponents of M, V and K in the semi-empirical model (Equations 19–24) are somewhere in between those in Equations (26–29). That is to say, the simpler models are some kind of bounds for the results of impact on a finite extent plate.

6. ONE PARAMETER NONDIMENSIONAL MODEL

The simplified model presented in this section is based on the approach first proposed by Zener [1], and later modified for composite laminates by Olsson [7]. The starting point for this model is Equation (A3) in the Appendix. In a previous work [15] the function $H(T)$ described by Equation (A4) was approximated by a linear function of T, and a one parameter model was derived. Such an approximation is found to be valid only for short impact durations ($T_{max} < 1$). In the present study, the function $H(T)$ is approximated by a quadratic function as:

$$H(T) = BT - (1/2M^*)T^2 \tag{30}$$

It should be noted that $H(T)$—unlike $G(T)$ defined by (A2) in the Appendix—is a rapidly converging series, and representation by a simple function as in Equation (30) is valid. The values of B and M^* can be found by least square curve fitting. The function $H(T)$ depends only on C and D. Hence for each pair of C and D, one can find a corresponding pair of B and M^*. Some examples are shown in

Figure 1. Using the parabolic approximation for H, Equation (A3) can be written as:

$$W(T) = B \int_0^T F(T^*)dT^* - (1/M^*) \int_0^T F(T^*)(T - T^*)dT^* \quad (31)$$

where B and M^* are constants to be determined. In Zener's approximation [1], the displacement of the plate at the point of application of force was proportional to the total impulse. This is represented by the first term on the right hand side of Equation (31). While considering impacts on finite extent plates, an apparent mass effect slows down the plate, and that is the physical meaning of the second term on the right hand side of Equation (31). Differentiating Equation (31) twice with respect to T, we obtain:

$$\ddot{W} = B\dot{F} - F/M^* \quad (32)$$

Following [1], Equations (14) and (32) can be reduced to a single equation as given below:

$$d^2Q/dT^2 + (1/M_e)F(Q) + BdF/dT = 0 \quad (33)$$

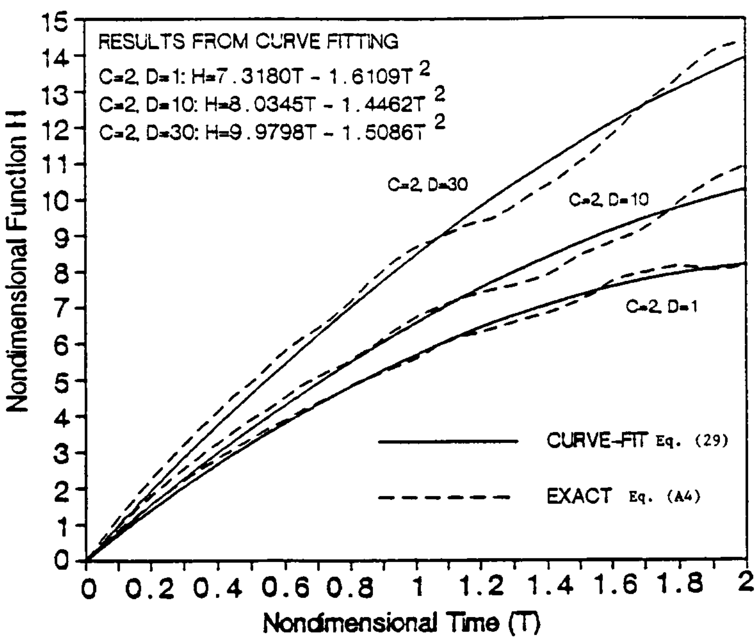

Figure 1. Comparison of exact and parabolic approximation of function H(T).

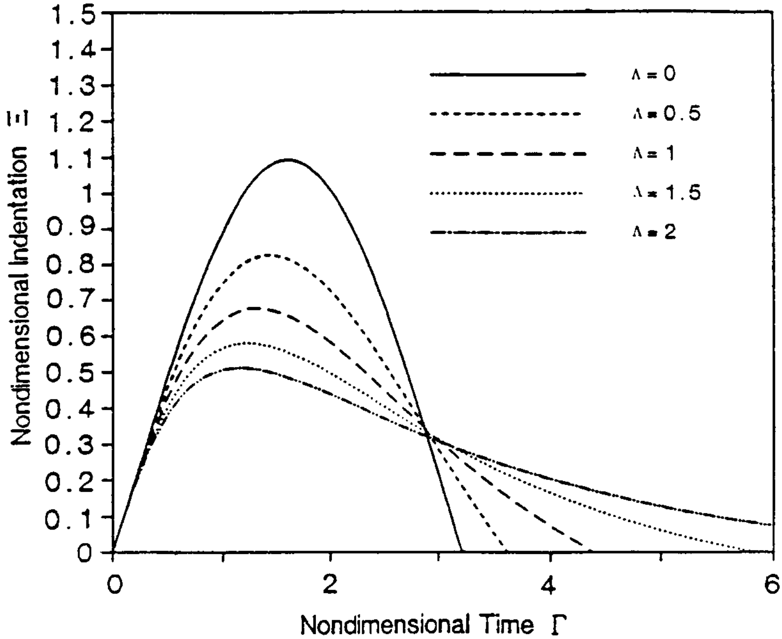

Figure 2. *Solution of the one-parameter nonlinear differential equation.*

where the equivalent mass M_e is given by

$$(1/M_e) = (1/M - 1/M^*) \tag{34}$$

The difference between the one parameter models derived in References [1,7,15] and the present model is that the second order effects due to the plate mass are included using the term M^*. As seen from Equation (34) the effect of M^* is to increase the effective mass M_e. The initial conditions for the nonlinear differential Equation (33) are: $Q = 0$, and $dQ/dT = 0$, at $T = 0$. The impact force F can be expressed in terms of Q using the contact law $F = KQ^{1.5}$. Then Equation (33) transforms into a nonlinear ordinary differential equation with a single parameter Λ that replaces all the five nondimensional parameters as shown below:

$$d^2\Xi/d\Gamma^2 + (1 + \Lambda d/d\Gamma)\Xi^{1.5} = 0 \tag{35}$$

where $\Xi = Q/\tau V$, $\Gamma = T/\tau$, $\Lambda = BM_e/\tau$, and $\tau = (M_e/KV^{0.5})^{0.4}$. Equation (35) can be solved numerically for various values of Λ as shown in Reference [1]. From the maximum value of Ξ, Q_{max} and hence F_{max} can then be calculated. The solutions of Equation (35), i.e., Ξ versus Γ, for various values of Λ are shown in Figure 2.

7. CONCLUDING REMARKS

A method for nondimensionalizing the impact equations is suggested in this paper. The particular method chosen in this study enables defining the impact problem in terms of five dimensionless parameters. The empirical formulas for maximum impact force and impact duration show that the simple spring-mass model and the half-space impact provide bounds for the actual force and impact duration. The nondimensional model can be used to estimate the impact response quickly during preliminary design of composite structures. In deriving the empirical formulas, the nondimensional parameters C and D are chosen such that the results are applicable to a wide variety of composite plates. Since the time dimension is normalized with respect to the fundamental period, problems involving large impact durations (as in the case of very flexible targets) can be solved in relatively smaller number of nondimensional time steps. If the impact duration is small, then the function H, which is the plate response to a unit step force, can be approximated by a simple quadratic polynomial, and the whole impact problem can be cast into a one parameter nonlinear differential equation.

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APPENDIX

In this section a numerical algorithm for solving Equations (13–15) is described. Although the method is developed for the system of nondimensional equations, they are equally applicable to Equations (1–3) also. Equation (13) can be written as:

$$W(T) = \int_0^T F(T^*)G(T - T^*)dT^* \tag{A1}$$

where the function G is given as:

$$G(T - T^*) = 4\Sigma_{m=1,3,\ldots}\ \Sigma_{n=1,3,\ldots}\ (1/\Omega_{mn}) \sin \Omega_{mn}(T - T^*) \tag{A2}$$

The function $G(T)$ represented by a double series in Equation (A2) is conditionally convergent, and hence cannot be represented as a simple function of T. Hence, we will rewrite (A1) as:

$$W(T) = - \int_0^T F(T^*)dH(T - T^*) \tag{A3}$$

where

$$H(T - T^*) = 4\Sigma_{m=1,3,\ldots}\ \Sigma_{n=1,3,\ldots}\ [1 - \cos \Omega_{mn}(T - T^*)]/\Omega_{mn}^2 \quad \text{(A4)}$$

From (A3) it is obvious that $H(T)$ is the center deflection $W(T)$ due to a unit step force. Integrating (A3) by parts and noting $F(0) = 0$,

$$W(T) = \int_0^T H(T - T^*)\dot{F}dT^* \quad \text{(A5)}$$

Let us assume that the variation of $F(T)$ during a small finite time interval ΔT is linear so that $\dot{F}$, the time derivative of F, is a constant during the interval, i.e.,

$$\dot{F} = A_i,\ (i - 1)\Delta T < T < i\Delta T \quad \text{(A6)}$$

From (A5) and (A6) we obtain:

$$W(i\Delta T) = \sum_{j=1}^{i} A_j S_1(i - j) \quad \text{(A7)}$$

where the function S_1 is defined as:

$$S_1(i - j) = \int_{(j-1)\Delta T}^{j\Delta T} H(i\Delta T - T^*)dT^* \quad \text{(A8)}$$

In order to understand the significance of S_1 consider the plate response due to a unit slant-step function $R(T)$ defined as:

$$\begin{aligned} R(T) &= T/\Delta T,\ 0 < T < \Delta T \\ &= 1,\ T > \Delta T \end{aligned} \quad \text{(A9)}$$

Substituting $R(T)$ for $F(T)$ in (A5), we obtain the response as:

$$W(T) = (1/\Delta T) \int_0^{\Delta T} H(T - T^*)dT^* \quad \text{(A10)}$$

By setting $j = 1$ in (A8), we obtain:

$$S_1(i - 1) = \int_0^{\Delta T} H(i\Delta T - T^*)dT^* \quad \text{(A11)}$$

Comparison of (A10) and (A11) shows that $(1/\Delta T)S_1(i - 1)$ is equal to the deflection at $T = i\Delta T$ due to a unit slant-step function $R(T)$. This physical meaning of the function S_1 will be useful in computing S_1, when numerical methods, such as finite elements, are used to model the impacted structure.

The impactor displacement $X(T)$ can be derived as:

$$X(i\Delta T) = Vi\Delta T - \sum_{j=1}^{i} A_j S_2(i - j) \tag{A12}$$

where the function S_2 is defined as:

$$S_2(i - j) = [(\Delta T)^3/6M][3(i - j)^2 + 3(i - j) + 1] \tag{A13}$$

In the numerical procedure we will satisfy the contact relation (15) exactly at the end of each time step. Thus for the ith time step:

$$\begin{aligned}(F_i/K)^{(1/\alpha)} &= Q_i \\ &= X_i - W_i \\ &= Vi\Delta T - \sum_{j=1}^{i} A_j[S_2(i - j) + S_1(i - j)]\end{aligned} \tag{A14}$$

Equation (A14) is a nonlinear algebraic equation in A_i and can be solved using a simple iterative scheme [11].

The above described numerical scheme is actually a modification as well as a more formal representation of the algorithm described in Reference [11]. It may be noted that the Function S_1 in the present study plays the role of S-function in Reference [11]. Further, S_1 depends only on the two nondimensional parameters C and D, and hence describe the plate response characteristics completely. They can be used with various combinations of M, V and K, which describe the impactor characteristics. Once the force history is determined in terms of A_i's, it can be substituted in Equations (16) and (17) to obtain the strain histories.

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