1 Introduction

Many microelectromechanical systems (MEMS) based electroacoustic devices, such as microphones, utilize composite plates [1–3]. The plate’s mechanical response to acoustic pressure oscillations is important in determining the sensitivity, bandwidth, and dynamic range of the electroacoustic device. The mechanical sensitivity of the device is dependent upon the plate compliance. The bandwidth is the frequency range over which the sensitivity is approximately constant. The resonance of the diaphragm determines the upper limit of the bandwidth. The dynamic range is the difference between the minimum detectable pressure and the maximum pressure where the sensitivity is independent of applied pressure.

In a piezoresistive MEMS microphone; for example, a silicon dioxide layer thermally grown on a silicon substrate provides electrical passivation and a silicon nitride layer deposited on top of the oxide serves as a hydrophobic barrier. The thermally grown oxide contains compressive stress and the deposited nitride layer can be tailored to be in tension or compression. A compressive stress increases the plate’s sensitivity to pressure and lowers the pressure at which the onset of nonlinear deflection occurs [3]. A sufficiently large compressive stress buckles the device. In contrast, a tensile stress lowers the sensitivity of the device while increasing the upper limit of the dynamic range [4]. Due to the significant impact of residual stresses on device performance, they must be incorporated into the model used in the design process.

Optimization has been shown to be an important tool in electroacoustic device design [5]. The performance tradeoff between the device sensitivity and noise floor suggests that the minimum detectable pressure is an appropriate objective function in microphone design optimization [5]. Performance, fabrication, and model accuracy constraints are required in the optimization. Two performance constraints are the bandwidth and the upper limit of the dynamic range. Linear plate theory is used to determine the mechanical sensitivity and bandwidth, but a nonlinear plate model is needed to predict the upper limit of the dynamic range.

The large deflection theory for isotropic, clamped plates with zero in-plane load is discussed in Timoshenko and Woinowsky-Krieger [6]. Sheplak and Dugundji [4] extended this work to nonlinear deflections of isotropic plates with tensile in-plane loading. The current effort builds upon the previous work by incorporating composite layers in the diaphragms. The structure analyzed is representative of a MEMS microphone. The plate is composed of a silicon-silicon dioxide-silicon nitride composite with material parameters shown in Table 1. This analysis, however, can be tailored for any number of composite layers, residual stresses, and material properties.

The derivation presented considers an axisymmetric plate and assumes transversely isotropic layers. The approximation of silicon as mechanically isotropic is common for MEMS structures due to its low degree of anisotropy [7,8]. The strain-displacement relationship implemented includes the von Kármán strains to accurately predict large deflections. Residual stresses are assumed to be uniform in each plate layer. The derivation results in two nonlinear, coupled, ordinary differential equations governing transverse deflection and the radial force resultant. The equations are nondimensionalized and solved using an iterative finite difference method. Results are reported for different combinations of nondimensional parameters that capture the composite lay-up and

### Table 1 Material and geometric parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Si</th>
<th>SiO₂</th>
<th>Si₃N₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>Layer thickness (µm)</td>
<td>10</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Young’s modulus (GPa)</td>
<td>170</td>
<td>73</td>
<td>384</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>0.27</td>
<td>0.17</td>
<td>0.24</td>
</tr>
</tbody>
</table>

*The diaphragm radius is 1.47 mm with a total diaphragm thickness h = 10.6 µm.*
The impact of in-plane tension on mechanical sensitivity and the onset of nonlinearity are discussed.

2 Basic Equations

Consider the cross section of an axisymmetric, composite plate shown in Fig. 1. A constant tensile in-plane stress, \( \sigma_0 \), is assumed to exist in one or more of the layers. A uniform transverse load, \( p \), is applied on the surface. Although only three layers are shown, the solution methodology is applicable to an arbitrary number of layers. The equilibrium equations of the axisymmetric plate are [9]:

\[
\frac{dN_r}{dr} + \frac{N_r - N_0}{r} = 0
\]  
(1)

\[
\frac{d}{dr} \left( \frac{dw}{dr} N_r \right) + rp + \frac{d}{dr} (rQ_r) = 0
\]  
(2)

and

\[
\frac{dM_r}{dr} + M_r - M_0 = Q_r
\]  
(3)

where \( N_r \) and \( N_0 \) are the radial and tangential in-plane force resultants, \( M_r \) and \( M_0 \) are the radial and tangential bending moment resultants, \( Q_r \) is the shear force resultant, and \( w \) is the deflection in the positive \( z \) direction.

Assuming an axisymmetric plate, the radial and tangential Green strain tensor simplified for small strains but moderate rotations in terms of displacement fields, \( u_r \) and \( u_z \), are [6]

\[
\begin{bmatrix} e_r \\ e_\theta \\ e_z \end{bmatrix} = \begin{bmatrix} \frac{\partial u_r}{\partial r} + \frac{1}{2} \left( \frac{\partial u_z}{\partial r} \right)^2 \\ \frac{\partial u_\theta}{\partial r} \\ \frac{\partial u_z}{\partial r} - z \frac{\partial w(r)}{\partial r} \end{bmatrix}
\]  
(4)

The Kirchhoff hypothesis yields the displacement field for a thin circular plate under uniform transverse loading is [9]

\[
u_z(r, z) = w(r)
\]  
(5)

and

\[
u_r(r, z) = \nu_0(r) - z \frac{\partial w(r)}{\partial r}
\]  
(6)

where \( u_0(r) \) is the radial and \( w(r) \) is the vertical displacement of points on the reference surface \( (z = 0) \) shown in Fig. 1. Substituting the displacement fields into Eq. (4), the strain relationships become [10]

\[
\begin{bmatrix} e_r \\ e_\theta \\ e_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial w}{\partial r} \end{bmatrix} + z \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_z \end{bmatrix}
\]  
(7)

where the reference plane strains are

\[
\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix} = \begin{bmatrix} \frac{du_r}{dr} + \frac{1}{2} \left( \frac{dw}{dr} \right)^2 \\ \frac{du_\theta}{dr} \\ \frac{du_z}{dr} - \frac{1}{r} \frac{dw}{dr} \end{bmatrix}
\]  
(8)

and the curvatures are

\[
\begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_z \end{bmatrix} = \begin{bmatrix} \frac{d^2w}{dr^2} \\ -\frac{1}{r} \frac{dw}{dr} \\ \frac{1}{r} \frac{d^2w}{dr^2} \end{bmatrix}
\]  
(9)

The plate is assumed to be transversely isotropic. The constitutive relationship for an axisymmetric plate including the residual stress is

\[
\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_r^0 \\ \sigma_\theta^0 \\ \sigma_z^0 \end{bmatrix} + [Q_n] \begin{bmatrix} \sigma_r^0 \\ \sigma_\theta^0 \\ \sigma_z^0 \end{bmatrix} + z [Q_n] \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_z \end{bmatrix}
\]  
(10)

where \( \sigma_r^0 \) is the residual stress in the \( n \)th layer and \( [Q_n] \) is the stiffness matrix of the \( n \)th layer.

\[
[Q_n] = \frac{E_n}{1 - \nu_n^2} \begin{bmatrix} 1 & \nu_n \\ \nu_n & 1 \end{bmatrix}
\]  
(11)

Here, \( E_n \) and \( \nu_n \) are the modulus of elasticity and Poisson’s ratio in the plane of the plate, respectively. The force and moment resultants are found by taking the zeroth and first moments of the stresses through the thickness of the plate, respectively, and are

\[
\begin{bmatrix} N_r \\ N_0 \\ M_r \\ M_0 \end{bmatrix} = \int_{z_a}^{z_T} \begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix} dz = \begin{bmatrix} N_0 \\ N_0 \\ M_0 \\ M_0 \end{bmatrix} + [A] \begin{bmatrix} \sigma_r^0 \\ \sigma_\theta^0 \\ \sigma_z^0 \end{bmatrix} + [B] \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_z \end{bmatrix}
\]  
(12)

and

\[
\begin{bmatrix} w \\ w \\ z_T \end{bmatrix} = \int_{z_a}^{z_T} \begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix} dz = \begin{bmatrix} M_0 \\ M_0 \\ \int_{z_a}^{z_T} [Q_n] \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix} dz \end{bmatrix}
\]  
(13)

where \( z_T \) and \( z_B \) are coordinates identifying the top and bottom of the plate. The \([A]\), \([B]\), and \([D]\) matrices are the extensional, flexural-extensional coupling, and flexural stiffness matrices, respectively, and are defined as [10],

\[
\begin{bmatrix} A \\ B \\ D \end{bmatrix} = \int_{z_a}^{z_T} [Q_n] \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix} dz
\]  
(14)

Equations (1)–(13) are combined to form a differential equation in terms of deflection slope and radial force resultant. First, Eq. (2) is integrated,

\[
\frac{d}{dr} (rN_r) + \frac{r}{z} p + Q_r = 0
\]  
(15)

and combined with Eq. (3) to yield
Substituting Eq. (13) into Eq. (16) results in

\[ \frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d}{dr} \left( \frac{dw}{dr} \right) - \frac{N_0 dw}{D_D} + \frac{1}{D_D} \left( B_{12} \frac{1}{2} \frac{dw}{dr} + N_t \right) + \frac{r}{2D_D} = 0 \]  

(17)

where \( \tilde{N}_t = N_t - N_0 \), \( B_{12} = B_{12}(A_{12}/A_{11}) - B_{12}, \) and \( D_s = D_{11} - (B_{11}/A_{11}) \). By introducing the transverse deflection slope (or approximate rotation angle of the transverse normals), \( \phi = -d w / dr \), Eq. (17) becomes

\[ \frac{d^2 \phi}{dr^2} + \frac{d}{dr} \left( \frac{\phi}{r} \right) - \frac{N_0}{D_D} \phi = \frac{\phi \tilde{N}_t}{D_D} + \frac{B_{12} \phi^2}{2D_D} - \frac{r}{2D_D} \]  

(18)

Equation (18) is a nonlinear differential equation with two unknowns, \( \phi \) and \( N_t \). Therefore, another equation with \( \phi \) and \( N_t \) as dependent variables is required. The second equation is derived from the strain compatibility equation [6].

\[ \varepsilon_t = \frac{d}{dr} \left( r \varepsilon_0 \right) + \frac{1}{2} \phi^2 \]  

(19)

Solving Eq. (12) for the radial and tangential strains and substituting into Eq. (19) results in an equation in terms of \( \phi \) and \( N_t \).

\[ r^2 \frac{d^2 \tilde{N}_t}{dr^2} + 3r \frac{d \tilde{N}_t}{dr} = -B_{12} \left( r \frac{d \phi}{dr} + \frac{\phi}{r} \right) - \frac{A_{12} - A_{11} \phi^2}{A_{11}} \]  

(20)

Equations (18) and (20) form two mixed force and displacement, nonlinear differential equations in dependent variables \( \phi \) and \( N_t \). Finally Eq. (12) is substituted into Eq. (1), yielding

\[ \frac{d^2 u_0}{dr^2} + \frac{1}{r} \frac{d u_0}{dr} - \frac{u_0}{r^2} = \frac{B_{11} \phi^2}{A_{11}} \left( \frac{d^2 \phi}{dr^2} + \frac{2 \phi}{r} \right) - \left( \frac{1}{A_{11}} \right) \frac{a^2}{k^2} \phi^2 - \phi \frac{d \phi}{dr} \]  

(21)

a nonlinear ordinary differential equation for the radial displacements along the reference surface. Equations (18), (20), and (21) are nondimensionalized using the following definitions

\[ \zeta = \frac{r}{a}, \quad \eta = \frac{a}{h}, \quad W = \frac{w}{R}, \quad \Theta = -\frac{d W}{d \zeta} = \frac{a}{h} \phi, \quad U = \frac{u_0}{h}, \quad S^*_r = \frac{\xi_0 a^2}{D^*_r}, \quad k^* = \frac{N_0 a^2}{D^*_r}, \quad \text{and} \quad P^* = \frac{p a^4}{2 h D^*_r} \]  

where \( a \) and \( h \) are the radius and total thickness of the plate, respectively. The nondimensionalization is similar to that of Sheplak and Dugundji [4]. It is important to note, however, that \( S^*_r, k^*, \) and \( P^* \) are influenced by the composite nature of the plate through the composite stiffness, \( D^*_r \). The resulting nondimensional equations are

\[ \frac{d^2 \Theta}{d \zeta^2} + \frac{1}{\zeta} \frac{d \Theta}{d \zeta} + \left( k^2 + \frac{1}{\zeta^2} \right) \Theta = -P^* \zeta + S^*_r \Theta - \frac{\Lambda}{2 \zeta} \Theta^2 \]  

(22)

and

\[ \zeta^2 \frac{d^2 \zeta^*}{d \zeta^2} + 3 \zeta \frac{d \zeta^*}{d \zeta} = -\Lambda \left( \frac{d^2 \Theta}{d \zeta^2} + \frac{d \Theta}{d \zeta} - \frac{\Theta}{\zeta} \right) - \frac{X}{2} \Theta^2 \]  

(23)

The \( \Lambda \) coefficient increases in magnitude as the plate becomes more asymmetric and goes to zero for a symmetric composite. The \( X \) coefficient is dependent upon the variation in the material properties of the composite layers. The \( \eta_{nu} \) coefficient is the nondimensional distance from the reference axis to the neutral axis. It is important to note that Eq. (22), Eq. (23), and Eq. (24) reduce to the equations of Sheplak and Dugundji [4] for a homogenous plate with \( z = 0 \) at its midplane.

The stresses within the plate Eq. (10) are nondimensionalized as

\[ \left\{ \begin{array}{c} \Sigma_\theta \\ \Sigma_0 \end{array} \right\} = \frac{1}{E_{nu} \zeta^2} \left( \begin{array}{c} \sigma_\theta^* \\ \sigma_0^* \end{array} \right) \]  

(25)

where \( E_{nu} \) is the chosen modulus of elasticity for nondimensionalization. The nondimensional stress can be decomposed into residual stress and stress due to transverse loading,

\[ \left\{ \begin{array}{c} \Sigma_\theta^* \\ \Sigma_0^* \end{array} \right\} = \left( \begin{array}{c} \Pi \\ 1 - \nu_\theta \end{array} \right) \left( \begin{array}{c} 1 \frac{d U}{d \zeta^2} + U \frac{\Theta}{\zeta} \\ \nu_\theta \frac{d U}{d \zeta^2} + U \frac{\Theta}{\zeta} \end{array} \right) + \eta \left( \begin{array}{c} \frac{d \Theta}{d \zeta^2} + \nu_\theta \frac{\Theta}{\zeta} \\ \nu_\theta \frac{d \Theta}{d \zeta^2} + \frac{\Theta}{\zeta} \end{array} \right) \]  

(26)

and \( \Pi = \frac{E_{nu}}{E_{nu} \zeta^2} \).

### 2.1 Boundary Conditions

The plate is assumed to have clamped boundary conditions such that the slope and radial displacement is zero at the plate’s edge,

\[ \Theta(\zeta = 1) = 0 \]  

(28)

and

\[ U(\zeta = 1) = 0. \]  

(29)

The symmetry condition dictates that the deflection slope and radial displacement is zero at the plate’s center,

\[ \Theta(\zeta = 0) = 0 \]  

(30)

and

\[ U(\zeta = 0) = 0 \]  

(31)
Two boundary conditions on $S_r^c$ are also needed. The symmetry condition also dictates that the radial change in the incremental radial force resultant is zero at the plate’s center,

$$\frac{dS_r^c}{d\zeta}|_{\zeta=0} = 0 \quad (32)$$

The second boundary condition on $S_r^c$ is derived from the clamped radial displacement boundary condition, Eq. (29). Combining Eq. (29) with Eqs. (1), (8), and (12), gives the second boundary condition,

$$\frac{dS_r^c}{d\zeta}|_{\zeta=1} + \left(1 - \frac{A_{12}}{A_{11}}\right)S_r^c|_{\zeta=1} = -\Lambda \frac{d\Theta}{d\zeta}|_{\zeta=1} \quad (33)$$

The deflection is obtained by integrating the slope $\Theta$ and applying a clamped boundary condition on the edge of the plate,

$$W(\zeta = 1) = 0 \quad (34)$$

### 2.2 Linear Theory.

For small deflections, the nonlinear terms in Eq. (22) are neglected and the resulting governing equation is uncoupled from $S_r^c$,

$$\frac{d^2 \Theta}{d\zeta^2} + \frac{1}{\zeta} \frac{d\Theta}{d\zeta} - \left(k^2 + \frac{1}{\zeta^2}\right)\Theta = -P^* \xi \quad (35)$$

The slope, transverse deflection, and curvature solutions are

$$\Theta(\zeta) = \frac{P^*}{k^2} \left(\zeta - \frac{I_1(k^*\zeta)}{I_1(k^*)}\right) \quad (36)$$

$$W(\zeta) = \frac{P^*}{k^2} \left\{\frac{(1 - \zeta^2)}{2} + \frac{1}{k^2} \frac{I_0(k^*)}{I_1(k^*)} \left[I_0(k^*\zeta) - I_0(k^*)\right]\right\} \quad (37)$$

and

$$\Psi(\zeta) = \frac{P^*}{k^2} \left(1 - \frac{k^2 I_0(k^*)}{I_1(k^*)} + \frac{I_1(k^*\zeta)}{I_1(k^*)}\right) \quad (38)$$

respectively, where $I_0$ and $I_1$ are modified Bessel functions of the first kind and the nondimensional curvature is defined as $\Psi = d\Theta/d\zeta$. Neglecting the nonlinear terms in Eq. (24) and substituting in Eq. (36) yields the radial displacement

$$U = \frac{\eta_{na} P^*}{k^2} \left(\frac{I_1(k^*\zeta)}{I_1(k^*)} - \zeta\right) \quad (39)$$

The slope, deflection, and curvature results are plotted in Fig. 2. These results are similar in nature to the isotropic case [4] with the effect of the composite accounted for in $k^*$ and $D^*$. The linear theory for a composite differs from the isotropic case by having a finite radial displacement. It is easily seen from Eq. (39) that in the limit of a homogenous plate evaluated at the midplane, i.e., $\eta_{na} \to 0$, the radial strain is identically zero. As the tension parameter, $k^*$, becomes large, the transverse deflection's inflection point (where curvature equals zero) gets closer to the edge of the plate and the deflection slope gradients become severe. This is indicative of an edge zone that develops between the point of maximum deflection slope and the clamped boundary condition at the plate's edge. As the tension parameter increases, this zone becomes smaller and the gradient in the deflection becomes larger. Similar to the isotropic plate case, the extent of this edge zone is estimated by $D^* \approx 3/k^*$ [4]. In a numerical solution, the edge zone must be
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3 Nonlinear Theory

For large deflections, the nonlinear terms in equations Eq. (22) and Eq. (23) become important. The governing equations are discretized and solved using finite difference equations. First, the solution method of Sheplak and Dugundji [4] is followed by introducing a coordinate transformation to concentrate points at the plate’s edge where deflection slope gradients are large. The coordinate transformation to a uniformly discretized variable, $\xi$, is [11]

$$
\xi = \ln \left( \frac{\beta + \xi}{\beta - \xi} \right)
$$

(41)

where $\beta$ ($>1$) is the stretching parameter and $\xi$ is a set of evenly spaced points. As $\beta \to 1$, the transformation groups more points near $\xi = 1$. After applying the coordinate transformation, the coupled equations become

$$
\left( \frac{d^2 \xi}{d\zeta^2} \right) \frac{d^2 \Theta}{d\xi^2} + \frac{1}{\xi \zeta} \frac{d^2 \xi}{d\zeta^2} \frac{d^2 \Theta}{d\xi^2} + \frac{d^2 \xi}{d\zeta^2} \frac{d^2 \Theta}{d\xi^2} - \frac{1}{\xi^2 \zeta^2} \frac{d^2 \Theta}{d\xi^2} = -\Lambda \frac{d^2 \Theta}{d\xi^2} - \Theta
$$

(42)

and

$$
\frac{d^2 U}{d\zeta^2} \left( \frac{\partial^2 \xi}{d\zeta^2} \right)^2 + \frac{dU}{d\zeta} \left( \frac{\partial^2 \xi}{d\zeta^2} \right) + \frac{1}{\xi \zeta} \frac{dU}{d\xi} \frac{d^2 \xi}{d\zeta^2} - \frac{U}{\xi \zeta^2} = 0
$$

(43)

The transformed boundary conditions are

$$
\Theta(\xi = 0) = \Theta(\xi = 1) = 0
$$

(45)

$$
\left. \frac{d^2 \xi}{d\zeta^2} \frac{dS_\zeta}{d\xi} \right|_{\xi = 0} = 0
$$

(46)

$$
\left. \frac{d^2 S_\zeta}{d\zeta^2} \right|_{\xi = 1} + \left( 1 - \frac{A_{12}}{A_{11}} \right) S_\eta \mid_{\xi = 1} = -\Lambda \left. \frac{d \xi}{d\zeta} \frac{d\Theta}{d\xi} \right|_{\xi = 1}
$$

(47)

and

$$
U(\xi = 1) = U(\xi = 0) = 0
$$

(48)

Second-order finite difference equations are used to discretize Eqs. (42), (43), and (44). Third-order forward and backward

---

**Table 2 Non-dimensional parameters**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda$</td>
<td>Symmetry</td>
<td>0.0162</td>
</tr>
<tr>
<td>$X$</td>
<td>Composite</td>
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</tr>
<tr>
<td>$\eta_{\text{N-D}}$</td>
<td>N-D Distance to neutral axis</td>
<td>-0.4344</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Aspect ratio</td>
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</tr>
</tbody>
</table>

---

Fig. 3 Linear nondimensional stresses for various $k^*$ when the plate is loaded with $P^* = 1$
difference equations are used to discretize the boundary conditions. The resulting matrix equations are

\[
\frac{1}{2} F(S_r C^3 r g, f H g) = f J g \quad (49)
\]

\[
\frac{1}{2} K(S_r C^3 r g) = f L(f H g) g \quad (50)
\]

and

\[
[R] \{U\} = \{T(\{\Theta\})\} \quad (51)
\]

An implicit iterative scheme is used to solve equations, Eqs. (49) and (50). First, Eq. (50) is inverted and solved for \( S_r C^3 r g \) given an initial guess for \( H fg \). Next, \( S_r C^3 r g \) and the guess for \( H fg \) are inserted into \( F(S_r C^3 r g, f H g) \) and Eq. (49), is solved for a new \( \Theta \). The new \( \{\Theta\} \) is substituted into Eq. (50), and the process is repeated until the solution converges. The initial guess used in this study is the linear solution. The convergence tolerance was set at 0.1% with the stretching parameter \( \beta = 1.01 \). After a solution is found for \( \{\Theta\} \), Eq. (51), is solved for \( \{U\} \). Input parameters for the governing equations to solve for \( \{\Theta\} \) and \( \{S_r\} \) are \( k^* \), \( P^* \), \( \Lambda \), and \( X \). For this paper, \( \Lambda \) and \( X \) were kept at constant values given in Table 2 that are indicative of a silicon-silicon oxide-silicon nitride composite [3]. In addition to these values, the nondimensional distance to the neutral axis from the reference axis \( \eta_{nu} \) and aspect ratio \( \zeta \) of the plate are needed to solve for the radial displacement, \( U \). Their values are also found in Table 2.

The numerical plate model is validated using the commercial finite element package ABAQUS. An axisymmetric composite plate model meshed with seventy-four 3-node quadratic thin (or thick) shell elements (type S AX2) is used. Residual stress is supplied as an initial condition.

Figure 4 shows the deflection, deflection slope, curvature, and radial displacement for increasing values of \( P^* \) (for \( k^* = 0 \)). In the linear solution, the deflection slope and mode shapes are independent of pressure loading. It is interesting to note that as \( P^* \) increases, the plate tends to behave similarly to a linear plate with

Fig. 4 Nonlinear nondimensional solution for various pressure loadings \((k^* = 0)\). Finite element results are marked by a circle.

(a) Deflection normalized by center deflection, \( W_0 \).

(b) Deflection slope normalized by center deflection, \( W_0 \).

(c) Curvature normalized by \( P^* \).

(d) Radial displacement normalized by center deflection, \( W_0 \).

Fig. 5 Nonlinear nondimensional stresses for various \( k^* \) when the plate is loaded with \( P^* = 1 \).

(a) Radial and tangential stress as a function of \( \xi \) at the bottom of the plate.

(b) Radial stress as a function of \( \eta \) at the edge of the plate.
This model is verified via the commercial finite element analysis package ABAQUS, and was found to compute the solution $O(10^3)$ times faster. The model captures the tradeoff between mechanical sensitivity and the upper limit of dynamic range. By increasing the in-plane tension of the plate, the upper limit of the dynamic range is increased. This occurs at a detriment to the mechanical sensitivity of the device. This tradeoff must be considered when designing sensors and actuators that incorporate composite plates with tensile stress.

Acknowledgment
This work was supported in part by The Boeing Company. The work of M. Williams and B. Griffin was supported by National Science Foundation Graduate Fellowships.

Nomenclature

- $r[\xi]$ = radial coordinate (nondimensional)
- $z[\eta]$ = transverse coordinate (nondimensional)
- $a$ = plate radius
- $h$ = total plate thickness
- $z_T$, $z_B$ = top and bottom coordinates of the plate
- $u_r$ = radial displacement field
- $u_t$ = transverse displacement field
- $w_0(U)$ = radial displacement at reference plane
- $w(W)$ = transverse displacement (nondimensional)
- $\phi(\Theta)$ = deflection slope (nondimensional)
- $\psi$ = nondimensional curvature, $d\Theta/d\xi$
- $\varepsilon_r$ = radial strain
- $\varepsilon_\theta$ = tangential strain
- $\varepsilon_\phi$ = radial strain at the reference plane
- $\kappa_r$ = radial curvature term
- $\kappa_\theta$ = radial curvature term
- $\sigma_r(S_r)$ = radial stress (nondimensional)
- $\sigma_\theta(S_\theta)$ = tangential stress (nondimensional)
- $\sigma_0(S_0)$ = residual tensile stress (nondimensional)
- $[Q]$ = stiffness matrix
- $p[P^*]$ = uniform transverse load (nondimensional)
- $Q_s$ = shear force resultant
- $N_r(S_r)$ = radial force resultant (nondimensional)
- $N_\theta$ = tangential force resultant
- $N_0$ = residual tensile force resultant
- $k^*$ = tension parameter
- $K_r$ = incremental radial force resultant
- $M_r$ = radial moment resultant
- $M_\theta$ = tangential moment resultant
- $M_0$ = residual moment resultant
- $E(\Pi)$ = modulus of elasticity (nondimensional)
- $\nu$ = poisson’s ratio
- $[A]$ = extensional stiffness matrix
- $[B]$ = flexural-extensional matrix due to coupling
- $[D]$ = flexural stiffness matrix
- $B_{11} = B_{11} + A_{12}^2 - B_{12}$
- $D^* = D_{11} - B_{11}^2/A_{11}$
- $\Lambda = [B_r^* h/D^*]$
- $X = (A_{11} - A_{12})h^2/A_{11} D^*$
- $\xi = \text{aspect ratio, } \frac{h}{h_0}$
- $\xi_n = \text{non-dimensional distance from the reference axis to the neutral axis, } \frac{A_{11} h}{h_0}$
- $\tilde{\xi} = \text{transformed non-dimensional radial coordinate, } \frac{A_{11} h}{h_0}$
- $\tilde{\beta} = \text{stretching parameter, } \frac{h}{h_0}$
- $[F]$ = finite difference coefficient matrix of $\Theta$
- $[K]$ = finite difference coefficient matrix of $S_r^*$
- $[R]$ = finite difference coefficient matrix of $U$

References


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