EXP(X) AND ITS PROPERTIES

One of the most important mathematical constants is the irrational number e=exp(1)= 2.718281828459045. It first arose at the time of the invention of calculus by Isaac Newton (1642-1727) and Gottfried Wilhelm Leibnitz (1646-1716) when looking at the derivative of the function f(x)=a^x . Using the standard definition of a derivative one has-

$$f(x)'=\frac{d(a^x)}{dx}=\frac{\lim}{\Delta x\to 0}\left\{\frac{a^{x+\Delta x}-a^x}{\Delta x}\right\}=a^x\frac{\lim}{\Delta x\to 0}\left\{\frac{a^{\Delta x}-1}{\Delta x}\right\}$$

If we now set a=b+1 and use the binomial expansion we find-

$$a^{\Delta x}=(b+1)^{\Delta x}=1+b\Delta x+\frac{b^2}{2!}\Delta x(\Delta x-1)+\frac{b^3}{3!}\Delta x(\Delta x-1)(\Delta x-2)...$$

Now looking at the term in the large curly bracket on the right of the f(x)' expansion we find-

$$\frac{\lim}{\Delta x\to 0}\left\{\frac{a^{\Delta x}-1}{\Delta x}\right\}=(a-1)-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-...$$

But we recognize that this expansion just equals the infinite series for ln(a). When a=2 one recovers the Gregory formula for ln(2)=0.693147…<1 Also when a=3 we have ln(3)=1.098612>1. This implies there is a number 'a' in the range 2<a<3 where ln(a) becomes unity. This number is a=e=2.718281828459045… For it we have the important relation that-

$$\frac{d(e^x)}{dx}=e^x=\exp(x)$$

This is the only function of x which has its derivative equal to itself.

Let us look at some additional properties of exp(x).

We begin by giving a Maclaurin expansion of exp(x) about x=0. It reads-

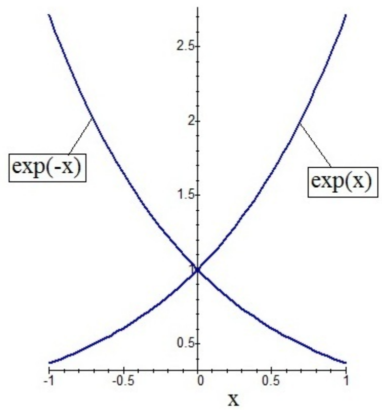
$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+..=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

So when x=1 we get the identity-

$$e=\exp(1)=\sum_{n=0}^{\infty}\frac{1}{n!}=1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\frac{1}{720}$$

This is a rather rapidly converging series because of the factorial term in the denominator of the series. A graph of exp(x) and exp(-x) follows-

THE EXPONENTIAL FUNCTIONS EXP(X) AND EXP(-X)



Note that the function exp(-x) has the same series form as exp(x) except that the odd powers of x become negative. Adding together exp(x) and exp(-x) we get a series with all positive signs . It reads-

$$\exp(x)+\exp(-x)=2\{1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+...\}+$$

Thus we have a new even function-

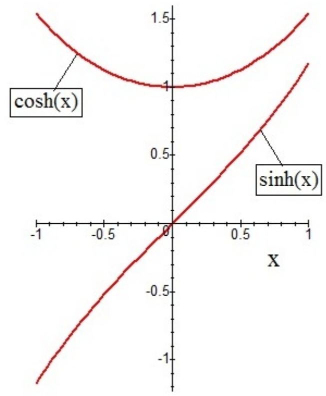
$$\cosh(x)=\frac{\exp(x)+\exp(-x)}{2}=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

known as the hyperbolic cosine function. It is an even function with a minimum value of one at x=0 and infinity as |x| approaches infinity. One also has a second hyperbolic function defined as-

$$\sinh(x)=\frac{\exp(x)-\exp(-x)}{2}=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

This function has odd symmetry since sinh(-x)=-sinh(x). A plot of both hyperbolic functions follows-

THE HYPERBOLIC FUNCTIONS SINH(X) AND COSH(X)



We can also combine these hyperbolic functions to yield-

exp(x)=cosh(x)+sinh(x)   and   exp(-x)=cosh(x)-sinh(x)

Replacing x by ix where i=sqrt(-1) we arrive at the famous Euler Identity-

$$\exp(ix) = \{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ...\} + i\{x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...\} = \cos(x) + i\sin(x)$$

This implies that –

$$\cosh(ix) = \cos(x) \quad and \quad \sinh(ix) = i\sin(x)$$

and also that-

Real{exp(ix)}=cos(x)   and   Imag{exp(ix)}=sin(x)

About six years ago we discovered a new way to get very accurate approximate values of trigonometric functions using integrals involving Legendre polynomials.(see http://www2.mae.ufl.edu/~uhk/TRIG-APPROX.pdf). This same technique can be used when dealing with the even hyperbolic function cosh(x/2). One notes that the integral-

$$\int_{x=-1}^{1} P[2n,x]\cosh(x/2)dx = N(n)\exp(1/2) - M(n)\exp(-1/2)$$

,where N(n) and M(n) are polynomials in n and P[2n,x] are the even Legendre polynomials of order 2n. Since the integral on the left has 2n zeros in the range -1<x<1 its value approaches zero as n gets large. Hence we have the approximation for exp(1) of –

$$e \approx M(n)/N(n)$$

If we now take n=10, we get the approximation-

e≈{1102315308988650200439441647042/405519139865470406785501069202}

=2.71828182845904523536028747135266249775724709369995957496696…

which is good to 60 decimal places. To get this accuracy with the above infinite series representation for exp(1) would require summing the first 47 terms of the series.

Several years ago we constructed a mnemonic for remembering exp(1) . It goes as follows-
e=
2.7+Andrew Jackson twice+right triangle+Fibonacci three+full circle+one year before crash-Boing jet-end of black death in Europe+route going west

That is 2.7-1828-1828-459045-235-360-28-747-1352-66

The mnemonic yields exp(1) to 33 places and thus exceeds what a hand calculator or a mathematical handbook can produce.

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