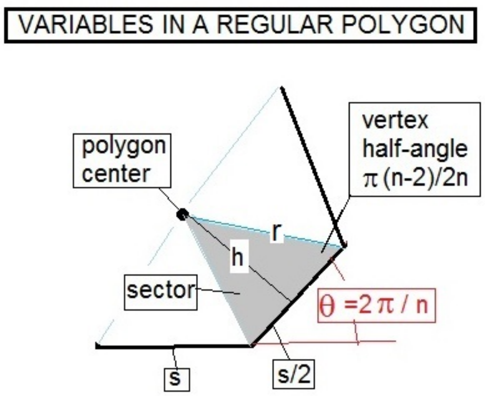
PROPERTIES OF REGULAR POLYGONS

The simplest 2D closed figures which can be constructed by the concatenation of equal length straight lines are the regular polygons including the equilateral triangle, the pentagon, and the hexagon. We want here to quickly derive some of the generic properties of such polygons including exterior and interior vertex angles, the length of their diagonals, and their area.

Our starting point is the following sketch of part of one of these polygons having n vertexes and side-length s-



We have an n sided regular closed polygon whose exterior angles must at up to 2π radians. Hence each of the exterior vertex angles equals-

$$\theta = 2\pi / n$$

This means that the interior vertex angle becomes-

$$\psi = \pi - \theta = \pi \frac{(n-2)}{n}$$

The result implies that the sum of the interior vertex angles can become quite large as n increases. For example an octagon(n=8) has the sum of the interior angles equal to 6π rad.

To calculate the area A(n) of an n sided regular polygon we need to just multiply the area of the gray sector shown in the figure by n. One finds the total polygon area to be-

$$A(n) = n\{sh/2\} = \frac{ns^2}{4}\tan(\frac{\psi}{2}) = \frac{ns^2}{4}\cot[\frac{\pi}{n}]$$

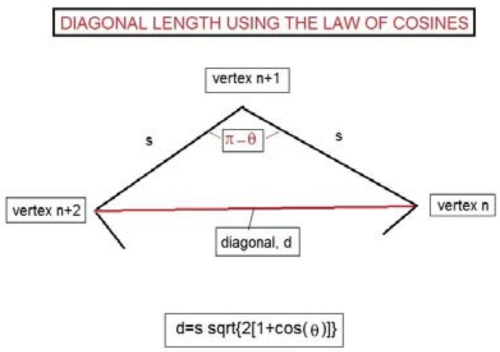
For the obvious case of a square (n=4) one has $A(2)=s^2$. For a hexagon (n=6) we get

$$A(6) = \frac{3\sqrt{3}}{2} s^2$$

As n goes to infinity we find $A(\infty)=\pi r^2$ as expected. From the geometry in figure 1 we also have-

$$h = r\sin[\frac{(\pi - \theta)}{2}] \quad and \quad s = 2r\cos[\frac{(\pi - \theta)}{2}]$$

We next look at the length of a diagonal line connecting vertex n with vertex n+2. Here we can make use of the law of cosines applied to the following triangle-



We get –

$$d^2 = 2s^2[1 - \cos(\pi - \theta)]$$

Hence-

$$d = s\sqrt{2(1 + \cos(\theta)}$$

For the case of a square we find d=s sqrt(2) , for a hexagon d=s sqrt(3), and for an octagon d=s sqrt[2+sqrt(2)].

An interesting result for d occurs when n=5. There-

$$(d/s) = \sqrt{2[1 + \cos(2\pi/5)]} = 2\cos(\frac{\pi}{5}) = \frac{1+\sqrt{5}}{2} = \varphi$$

by using a double angle formula for $\cos(\theta)$ and looking up the value for $\cos(\pi/5)$.

The quotient term in this result is just the Golden Ratio φ found well over two thousand years ago by the ancient Greeks. Its value to one hundred places reads-

φ=1.618033988749894848204586834365638117720309179805762862135448622705260462818902449707207204189391138…

Note that it is an irrational number and thus may be useful in the generation of prime numbers. For example, p=1618033988749 is a prime.

In playing with the equality-

$$\frac{d}{s}=\frac{1+\sqrt{5}}{2}=\varphi$$

one sees there exists a right triangle, known as Kepler's Triangle, with sides a=sqrt(φ), b=1 and hypotenuse c=φ. So one has the equality-

$$\varphi+1=\varphi^2 \text{ which says that } \frac{1}{\varphi}=\varphi-1$$

Additional irrational numbers can be generated by finding the diagonals to higher n polynomials. Take the case of n=7. Here we have-

$$(\frac{\mathrm{d}}{\mathrm{s}})=\sqrt{2[1+\cos(\frac{2\pi}{7})]}=2\cos(\frac{\pi}{7})$$
$$=1.8019377358048382524722046390148901023318383242637$$

to fifty places.

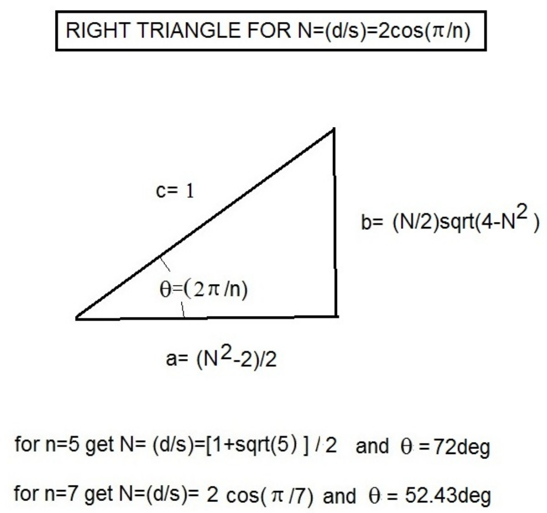
We can generalize the above diagonal lengths for any n sided regular polygon to the irrational number-

$$\mathrm{N(n)}=(\frac{d}{s})=2\cos(\frac{\pi}{n})$$

valid from n=4 through infinity. That is, the values for (d/s) lie in the range-

$$\sqrt{2}\le N(n)\le 2$$

Every n-sided regular polygon obeys the triangle rule shown-



There are an infinite number of irrationals N(n) possible. Some of these allow expression in terms of roots of integers. In particular we find the additional exact results-

N(10)=2 cos(π/10)=sqrt{[5+sqrt(5)]/2)
= 1.9021130325903071442328786667587642868113972682514…

and-

N(12)=2 cos(π/12)=[3+sqrt(3)]/sqrt(6)
=1.9318516525781365734994863994577947352678096780168…

These irrational numbers for the diagonal-side-length ratio are just as valid as is the Golden Ratio found at n=5. Note the approach toward N(∞)=2.

As a final consideration consider bounding the polygon area A(n) between the two circles of area πr^2 and πh^2, where r and h are the length indicated in the first figure above.
We can write that-

$$\pi h^2 < \frac{ns^2}{4}\cot(\frac{\pi}{n}) < \pi r^2$$

With a bit of mathematical manipulation this simplifies to the inequality-

$$\cos(\pi / n)^2 < \frac{n\sin(2\pi / n)}{2\pi} < 1$$

Hence we have –

$\pi > n\sin(2\pi/n)/2$ and $\pi < n\tan(\pi/n)$

As n->infinity, both terms go to-

π=3.14159265358979323846264338327950288419716939937510…

but they converge to this limit very slowly as Archimedes(287-212BC) already found out several thousand years ago. For a hundred-thousand sided regular polygon we still have only a six decimal place accuracy for π. Here is the inequality at n=100,000-

$$3.141592448<\pi< 3.141592757$$

A former colleague of mine here at the University og Florida was Dr.Karl Pohlhausen of boundary layer fame. He told me that when he was a child back in the late 18 hundreds he would recall the value of π by the mnemonic "Drei komma Huss Verbrandt". This meant 3 plus the year 1415 when the Czech religious reformer J. Huss was burnt at the stake.

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