## USING VECTORS TO MEASURE ANGLES BETWEEN LINES IN SPACE

Consider a straight line in Cartesian 3D space [x,y,z]. Let two points on the line be  $[x_1,y_1,z_1]$  and  $[x_2,y_2,z_2]$ . The slopes of this line are constants and read-

$$\frac{\partial z}{\partial x} = \frac{(z_2 - z_1)}{(x_2 - x_1)} = A \quad and \quad \frac{\partial z}{\partial y} = \frac{(z_2 - z_1)}{(y_2 - y_1)} = B$$

From these we can generate the equation for a straight line in 3D as-

$$A(x - x_1) = B(y - y_1) = (z - z_1)$$

If we now have the line pass through the origin then  $x_1, y_1$ , and  $z_1$  can be set to zero and we simply have Ax=By=z. Furthermore on letting the slope values A and B equal to one, we have a line coming out of the origin making equal angles of  $\cos^{-1}[1/\text{sqrt}(3)]=60.8173^{\circ}$  with respect to the x, y, and z axes. In a parametric representation, this last straight line is defined as-

$$x=y=z=t$$
 with -  $$ 

In general, for any straight line passing through the origin we can speak of its direction cosines given by cos(), cos() and cos(), where , ,and are the angles the line makes with respect to the x, y, and z axis.

A very convenient way to represent a straight line is by means of a constant vector defined as-

$$V=i(x_2-x_1)+j(y_2-y_1)+k(z_2-z_1)$$

Here the bold lettering indicates a vector having both magnitude and direction with  $\mathbf{i}, \mathbf{j}$ , and  $\mathbf{k}$  the unit length vectors along the x,y, and z axis, respectively. The magnitude of this vector is-

$$|\mathbf{V}| = \operatorname{sqrt}[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]$$

and thus represents just the distance between point  $[x_1,y_1,z_1]$  and  $[x_2,y_2,z_2]$ . One can find the cosine of the angle between two vectors  $V_1$  and  $V_2$  via the identity-

$$\cos(_{''}) = \left\{ \frac{V_1 \cdot V_2}{|V_1| |V_2|} \right\}$$

The right side of this expression is just the dot(or scalar) product of the two vectors normalized to unit length. By setting  $V_2$  equal to one of the unit base vectors **i**, **j**, or **k**, we

will generate the direction cosines of the line. Thus a straight line passing through the origin and containing a second point at [1,2,3] has the direction cosines-

$$\cos(r) = \frac{1}{\sqrt{14}}$$
,  $\cos(s) = \frac{2}{\sqrt{14}}$  and  $\cos(x) = \frac{3}{\sqrt{14}}$ 

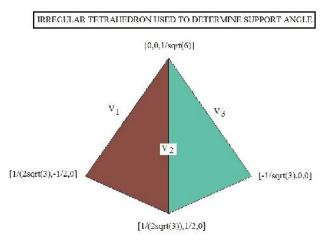
Notice that-

$$\cos(r)^{2} + \cos(s)^{2} + \cos(x)^{2} = 1$$

This will always be the case for any constant vector.

We are now in a position to determine the angles between any two lines in space. This is a very important problem in a variety of different areas including satellite tracking and determining the angle of certain saw cuts made in carpentry.

Let us begin the analysis by looking at a problem we encountered several years ago when constructing a wooden sculpture consisting of a cube mounted on a vertical support column such that the cube's diagonal was set vertical and hence was parallel to the support column. In that case one was asking for the angle between the vertical and a line bisecting one of the three square faces of the cube as seen when looking up from the column. To solve this problem we can turn things upside down and construct a computer generated irregular tetrahedron as shown-



The three vectors indicated in the figure represent the bottom three straight line edges of the cube as seen from the column support which is now at the apex of this tetrahedron. The sides of the tetrahedron represent 45-90-45 degree triangles while the base is an equilateral triangle of side-length sqrt(2). The coordinates of the vertexes are as indicated.

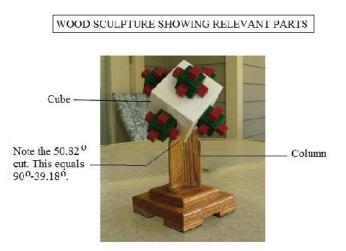
To now get the required angle for the cut required of the column to have a snug fit with the cube, we construct a fourth vector bisecting the brown face in the figure. This new vector reads-

$$V_4 = i(\frac{1}{2\sqrt{3}}) - k(\frac{1}{\sqrt{6}})$$

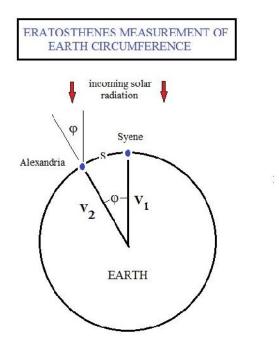
On performing a scalar multiplication with the negative z axis vector  $V_5$ =-k, we find that-

$$u_{\pi} = \cos^{-1}\left\{\frac{1/\sqrt{6}}{1/2}\right\} = \cos^{-1}\sqrt{\frac{2}{3}} = 39.18265^{\circ}$$

Thus the column support must have its three identical components placed at  $120^{\circ}$  intervals and have end-cuts of about  $39.18^{\circ}$ . It is difficult to see how this result could have been arrived at by any other means except by trial and error using cardboard and scissors as we first did when designing the sculpture. Here is a picture of the sculpture showing the relevant parts used in the above discussion-



Another problem, involving the angle between two vectors, deals with finding the circumference of the Earth. This problem was first looked at by Eratosthenes of Alexandria over two thousand years ago. He started with the correct assumption that the earth is essentially a sphere of radius R and circumference C=2 R. Then he noted that the sun was directly overhead at noon during the summer solstice in Syene, Egypt while at the same time at his home in Alexandria the sun's incoming radiation was at  $1/50^{\text{th}}$  of a full circle( corresponding to  $=7.2^{\circ}$ ) from the local zenith. It was also known that Syne lies some 500 miles south of Alexandria. From these facts one arrives at the following picture-



Here we have two lines connecting Alexandria and Syene with the Earth center. These lines can be represented by the vectors-

 $V_1 = kR$  and  $V_2 = -iRsin() + kRcos()$ 

, where R is the earth radius. From the geometry we also have that S=R  $\,$  . Thus we can conclude using Eratosthenes's measurements that the earth circumference is-

C=2 S/ =50.500=25,000 miles

This is an amazingly accurate result compared to the exact value of C=24,860 miles.

As a third problem involving the angle between straight lines consider finding the shortest distance between the parabola  $y=x^2$  and the line y=x-1. Here the vector defining a straight line perpendicular to the parabola is its gradient given by-

 $V_1$ =grad( $x^2$ -y)=2ix-j

Also the straight line x-y=1can be represented by the vector  $V_2=i+j$ .

Taking the dot product between  $V_1$  and  $V_2$  we find-

$$\cos(\{\) = \frac{(2x-1)}{\sqrt{8x^2+2}}$$

The shortest distance between the parabola and the straight line occurs when vectors  $V_1$  and  $V_2$  are orthogonal to each other so that cos()=0. That is , x=1/2 and y=1/4 on the

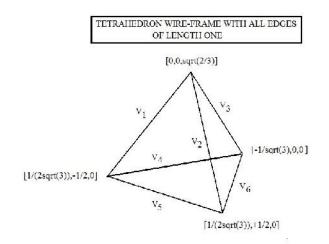
parabola. The equation for the line parallel to the gradient and passing through this point will be y+x=3/4. This last line intersects the original straight line at x=7/8 and y=-1/8. Hence the shortest distance between the parabola  $y=x^2$  and the line y=x-1 will be-

$$u = \sqrt{\left(\frac{7}{8} - \frac{1}{2}\right)^2 + \left(\frac{-1}{8} - \frac{1}{4}\right)^2} = \frac{3}{8}\sqrt{2} = 0.530330...$$

As the last example we wish to find the center of mass of a uniform density regular tetrahedron having four equilateral triangles of side length 1 each. Here we are dealing with a polyhedron of four sides S, four vertexes V, and six edges E. It obeys the famous Euler formula -

$$S+V-E=2$$

We begin our discussion for finding the center of mass by looking at the following wire frame of a tetrahedron-



The six vectors connecting the four vertexes are shown as well as the Cartesian coordinates of these vertexes. Explicitly we can write, for instance, that-

$$V_2 = (\frac{1}{2\sqrt{3}})i + (\frac{1}{2})j - (\sqrt{\frac{2}{3}})k \quad with \quad |V_2| = 1$$

The origin of the coordinate system has been placed at the centroid of a triangular lamina forming the tetrahedron base. Notice the [0,0,0] point is located where a vertical line emanating from the upper vertex intersect the bottom lamina. Each of the six edges of the tetrahedron have length 1 as can readily be established. From symmetry arguments one sees at once that the center of mass of the uniform density tetrahedron having the same dimensions as the wire-frame model must lie along the vertical axis passing through the coordinate origin. To calculate the mass center we need only evaluate the integral-

$$\overline{z} = \frac{A_0 \int_{z=0}^{b} z(1-\frac{z}{b})dz}{A_0 \int_{z=0}^{b} (1-\frac{z}{b})dz}$$

, where  $A_0$  is the area of the base equilateral triangle and b=sqrt(2/3) the height of the tetrahedron. The linear term (1-(z/b) allows for the fact that the triangle lamina decrease in area with increasing value of z. Although the base area  $A_0$  need not be evaluated explicitly to find zbar, it is easy enough to get by noting that this area is just half the cross-product between vectors  $V_5$  and  $V_6$ . We have-

$$A_{0} = \frac{1}{2} |V_{5}xV_{6}| = \frac{1}{2} abs \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \end{vmatrix} = \frac{\sqrt{3}}{4}$$

The evaluation of zbar as the quotient of two definite integrals is straight forward and yields-

$$\overline{z} = \frac{\sqrt{2/3}}{3} = (1/3) \cdot (tetrahedron \ height)$$

Thus the tetrahedron mass center lies at [0,0,(1/3)sqrt(2/3)]. This could have been anticipated considering the symmetry of the problem in the x and y directions. It would be interesting to see if the King's Chamber in the great Pyramid of Cheops at Giza is located at the mass center of this square base pyramid. The calculation for zbar should be identical despite of the fact that a true pyramid has a square and not a triangular base.

December 2013