A NEW METHOD FOR FINDING ACCURATE ESTIMATES FOR THE VALUES OF CERTAIN FUNCTIONS

About seven years ago (see http://www2.mae.ufl.edu/~uhk/EVAL-ARCTAN.pdf and http://www2.mae.ufl.edu/~uhk/IEEETrigpaper1.pdf) we showed that certain integrals based upon the product of rapidly oscillating Legendre polynomials when multiplied into certain slowly varying functions such as 1/(x²+a²) or sin(ax) can lead to excellent approximations for most trigonometric functions. We wish here to generalize the discussions by obtaining approximate values for certain other constants and functions.

We begin by looking at the following definite integral-

\[ J(n,a) = \int_{-1}^{1} P[n,x] f[a,x] \, dx \]

where \( P[n,x] \) represents the Legendre Polynomials for large \( n \) and \( f[a,x] \) any slowly varying function of \( x \) containing no zeros or infinities. An interesting fact, crucial to the present discussion, is that the integral of any Legendre Polynomials taken over the double range \(-1<x<1\) is always zero. The first twelve of these polynomials have the form-

<table>
<thead>
<tr>
<th>( n )</th>
<th>Legendre Polynomials ( P[n,x] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x )</td>
</tr>
<tr>
<td>2</td>
<td>( (3x^2-1)/2 )</td>
</tr>
<tr>
<td>3</td>
<td>( (5x^3-3x)/2 )</td>
</tr>
<tr>
<td>4</td>
<td>( (35x^4-30x^2+3)/8 )</td>
</tr>
<tr>
<td>5</td>
<td>( (63x^5-70x^3+15x)/8 )</td>
</tr>
<tr>
<td>6</td>
<td>( (231x^6-315x^4+105x^2-5)/16 )</td>
</tr>
<tr>
<td>7</td>
<td>( (429x^7-693x^5+315x^3-35x)/16 )</td>
</tr>
<tr>
<td>8</td>
<td>( (6435x^8-12012x^6+6930x^4-1260x^2+35)/128 )</td>
</tr>
<tr>
<td>9</td>
<td>( (12155x^9-25740x^7+18018x^5-4620x^3+315x)/128 )</td>
</tr>
<tr>
<td>10</td>
<td>( (46189x^{10}-109395x^8+90090x^6-30030x^4-3465x^2-63)/256 )</td>
</tr>
<tr>
<td>11</td>
<td>( (88179x^{11}-230945x^9+218790x^7-90090x^5+15015x^3-693x)/256 )</td>
</tr>
<tr>
<td>12</td>
<td>( (676039x^{12}-1939938x^{10}+2078505x^8-1021020x^6+225225x^4-18018x^2+231)/1024 )</td>
</tr>
</tbody>
</table>

The number of zeros in \(-1<x<1\) is exactly \( n \) and \( P(2n,x) \) are even and \( P(2n+1,x) \) are odd functions. These symmetry properties become important when the slowly varying function \( f[a,x] \) is an even or odd function.

Consider now multiplying \( P[n,x] \) by a slowly varying function \( f[a,x] \) and then integrating the result. As \( n \) gets large the oscillatory portion of the integrand will cancel the local positive area by the adjoining negative area contribution. As a result the integral \( J(n,a) \) will quickly approach zero as \( n \) gets large. One thus finds
\[ J(n,a) = \{M(n,a) - N(n,a)g(a)\} \approx 0 \]

where \( M \) and \( N \) are typically polynomial functions in powers of \( a \) and-

\[ g(a) = \int_{x=0}^{1} f[x, a] \, dx \]

Thus last integral can often be integrated exactly. A few examples of \( g(a) \) are-

\[ \ln(1 + \frac{1}{a}) = \int_{x=0}^{1} \frac{dx}{(x + a)} \]
\[ \frac{1}{a} \arctan\left( \frac{1}{a} \right) = \int_{x=0}^{1} \frac{dx}{(x^2 + a^2)} \]
\[ \frac{1}{a} \{1 - \cos(a)\} = \int_{x=0}^{1} \sin(ax) \, dx \]
\[ \frac{1}{a} \{\exp(a) - 1\} = \int_{x=0}^{1} \exp(ax) \, dx \]

Since \( J(n,a) \) is expected to vanish as the oscillation frequency of \( P(n,a) \) gets large, we have the approximation -

\[ g(a) \approx \frac{M(n,a)}{N(n,a)} \]

which should produce quite accurate estimates for the function \( g(a) \) at any desired value of \( a \) smaller than about one. To demonstrate this result consider the function \( g(a) = \int (a/(x^2+a^2), x=0..1) \) and let \( a=1/(2\text{-sqrt}(3)) \) and \( n=20 \). This produces the 35 digit accurate value for \( \pi \) of-

\[ \pi = 3.1415926535897932384626433832795028 \]

when using the Legendre polynomial \( P(20,x) \) for our oscillatory function. This function has a total of 10 zeros in \( 0<x<1 \) and the integral \( \int [P(20,x), x=0..1]=0 \).

Consider next the error function-

\[ \text{erf}(a) = \left( \frac{2}{\sqrt{\pi}} \right) \int_{x=0}^{1} \exp(-ax^2) \, dx \]
I remember encountering this function the first time in my undergraduate differential equations course taught by Professor Monroe Martin at the University of Maryland about sixty years ago. Martin was a great teacher who believed in the Socratic method of teaching by example. One day he called me in front of the class to obtain an estimate for the integral-

\[ \int_{x=0}^{1} \exp(-x^2)dx \]

which he had written on the blackboard. I came back at once with the answer 3/4 and then proceeded to show how it was done in those pre-calculator days. The idea was to simply expand \( \exp(-x^2) \) in the Taylor series \( 1-x^2/1!+x^4/2!-x^6/3!+\ldots \) and then integrate term by term yielding \( 1-1/3+1/10-1/42+=26/35=0.742857 \). The actual value is 0.7468241…From that point on Professor Martin called me Mr Geodesic. As you know a geodesic is the shortest distance between two points on a surface and so matches the meaning of my surname of Kurzweg, which when translated from the German, means short-way.

On expanding the above definition for \( \text{erf}(a) \), we find-

\[ \text{erf}(a) = \left( \frac{2a}{\sqrt{\pi}} \right) \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{n!(2n+1)} = \left( \frac{2a}{\sqrt{\pi}} \right) \left\{ 1 - \frac{a^2}{1!3} + \frac{a^4}{2!5} - \frac{a^6}{3!7} + \ldots \right\} \]

We can now use the present method to evaluate the integral-

\[ J(n,a) = \int_{x=0}^{1} P(n,x) \exp(-(ax)^2)dx \]

and then compare the results with those obtainable from the above series expansion for \( \text{erf}(a) \). Choosing \( n=8 \), we have-

\[ P(8, x) = (6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35x)/128 \]

which has four zeros in \( 0<x<1 \) and the property that its integral over the given range is zero. Here one finds that-

\[ N(a) = \frac{675675}{4096} - \frac{45045}{512} a^2 + \frac{10395}{512} a^4 - \frac{315}{128} a^6 + \frac{35}{256} a^8 \]

and-

\[ M(a) = \frac{675675}{2048} a + \frac{45045}{1024} a^3 + \frac{5775}{512} a^5 + \frac{93}{256} a^7 \]

This leaves us with –
At $a=1$ one gets the approximation $\text{erf}(1)=0.8427006104$ compared to the exact value of $0.8427007929$ obtained by summing the above infinite series for $\text{erf}(a)$. Thus we have six place accuracy. The smaller one makes 'a' the more accurate the answer will become. As an example we look at $a=0.1$ where our approximation yields the ten place accurate result $0.1124629160$. Also improvements will occur if $n$ is increased further.

Consider next the integral-

$$J(20,a) = \int_{0}^{1} P[20,a] \exp(ax)dx$$

Here the oscillatory Legendre function has ten zeros in $0<x<1$. On setting this integral to zero and $a=1$, we get the approximation-

$$\exp(1) \approx 2.71828182845904523536028747$$

which is accurate to 27 digits. I can recall these 27 digits by use of my own constructed mnemonic-

$2.7$+$Andrew$+$Jackson$+$twice$+$right$+$triangle$+$Fibonacci$+$three$+$full$+$circle$+$year$+$before$+$crash$+$Boing$+$jet.$

As a final example of the application of the present method to functions consider the hyperbolic functions $\tanh(a)$, $\sinh(a)$ and $\cosh(a)$. Approximations to these may be generated using the integral-

$$J(n,a) = \int_{0}^{1} P[n,x] \cosh(ax)dx \quad \text{for} \quad n >> 1$$

Upon setting this integral to zero for larger $n$ one finds the approximation-

$$\tanh(a) \approx \frac{M(n,a)}{N(n,a)}$$

Here the numerator is a $n$-$1$ order polynomial in 'a' and the denominator represents an $n$th order polynomial in a for fixed n. The $n=4$ and $n=8$ order approximations read-
\[ \tanh(a) \approx \frac{105a + 10a^3}{105 + 45a^2 + a^4} \]

and

\[ \tanh(a) \approx \frac{2027025a + 270270a^3 + 6930a^5 + 36a^7}{2027025 + 945945a^2 + 51975a^4 + 630a^6 + a^8} \]

Evaluating these results at \( a=1 \) produces the approximations \( 0.7615894 \) and \( 0.761594155955758 \). The exact value is \( \tanh(1)=0.7615941559557648 \). Thus the \( n=8 \) approximation is good to 13 places. Plotting the second of these approximations in the range \(-5< a <+5\) produces-

The results are essentially indistinguishable from the exact value of \( \tanh(a)/a \). The red curve resembles the witch of Agnesi Curve and the green one represents a transition curve between two different constant values. I remember as a graduate student at Princeton that such curves came in handy when describing things such as density changes across a shock wave.

Approximations for other hyperbolic functions in terms of \( \tanh(a) \) are easy to obtain by using the identities-
\[
\sinh(a) = \frac{\tanh(a)}{\sqrt{1 - \tanh(a)^2}} \quad \text{and} \quad \cosh(a) = \frac{1}{\sqrt{1 - \tanh(a)^2}}
\]

The \( P(8,x) \) approximation using these last two identities produces:

\[
\sinh(1) = 1.175201194 \quad \text{and} \quad \cosh(1) = 1.543080635
\]

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