ARCHIMEDES AND VAN CEULEN

One of the most important constants in mathematics is the ratio of circle circumference C=2πR to its diameter D=2R. Denoted by the letter π it is an irrational number whose value accurate to 100 digits reads-

π=3.1415926535897932384626433832795028841971693993751058209749445923 0781640628620899862803482534211706 79

The ancient Egyptians and Babylonian around 2000BC knew that π had a value near 3.14 from mechanical measurements. But it was not until around 200BC that the famous Greek mathematician Archimedes of Syracuse (287-212BC) put the search for π on a sound mathematical basis. It is our purpose here to discuss the details of his approach to finding π, show how the Dutch mathematician Ludolph van Ceulen (1540-1610AD) took the Archimedes approach to an extreme, and finally indicate a new method based on an integral involving Legendre polynomials to determine π much more efficiently. None of these approaches will involve arctan formulas or AGM methods, which are today's preferred methods for estimating the value of π. With modified versions of the AGM method of Gauss and use of supercomputers, one can today determine π to well over a trillion places of decimal.

ARCHIMEDES APPROCH FOR FINDING PI:

The basic idea behind Archimedes's approach for estimating the value of π is to look at the areas of a circumscribing circle of radius R and the area of an inscribed circle of radius r and then compare things with a $3(2^n)$sided regular polygon lying in the annular region between the two circles. Starting with a simple hexagon, where the side number is $3(2^1)=6$, one finds-

$$3\sin(\frac{\pi}{3}) < \pi < 3(\frac{\sin(\pi/3)}{\cos(\pi/6)^2})$$

Although he was not familiar with trigonometry, one can easily show via the Pythagorean Theorem that sin(π/3)=1/2 and cos(π/6)=sqrt(3)/2. Hence we have-

$$2.598\ldots<\pi<3.464\ldots$$

This is thus seen to bracket π but does so rather poorly. To improve things he next took a twelve sided $3(2^2)$ regular polygon getting-

$$6\sin(\frac{\pi}{6}) = 3 < \pi < 6(\frac{\sin(\pi/6)}{\cos(\pi/12)^2}) = 3.21$$

This bracket is an improvement over the n=0 case, but still not what is desired. Notice by going from a $3(2^n)$ to a $3(2^{n+1})$ sided polygons in the analysis, one only needs to work out the half angle found in the denominator of the upper bound. This is possible without trigonometry using multiple triangles. Generalizing the Archimedes result for $3(2^n)$ sided regular polygons one obtains the brackets-

$$3(2^n)\sin[\frac{\pi}{3(2^n)}] < \pi < 3(2^n)\frac{\sin[\pi/(3(2^n)]}{\cos[\pi/(3(2^n)]^2}$$

Very accurate values of sqrt(3) must be knownin order to carry out these calculations. Archimedestook his calulations from 6 to 12 to 24 to 48 and finally to 96 sided regular polygons . The 96 sided polygon left him with the brackets-

$$3.1393<\pi<3.1427$$

In terms of fractions this reads-

$$3+(10/71)<\pi<3+1/7$$

Although this result barely reaches two decimal point accuracy, the approach is sound and will lead to any desired accuracy when considering a large enough sided polygons. What still amazes me is how Archimedes was able to obtain this result for n=5 which required both high root accuracy but also working out double angles. It is interesting that the 96 polygon result of $\pi \approx 3\ 1/7$ was still being taught in schools some 2000 years later. This was the case in my own middle school math class. Here is a short table using the Archimedes approach for regular polygons generated from a hexagon and going up to $3(2^{10})=3072$ sides-

Number of Polygon Sides	Lower Bound for Pi	Upper Bound for Pi
6	3.000…	3.215…
12	3.1058…	3.1596…
24	3.14608…	3.13262…
48	3.139350…	3.142714…
96	3.1410319…	3.1418730…
192	3.14166274…	3.14145247…
384	3.141610178…	3.141557609…
768	3.141597034…	3.141583892...
1536	3.141593750	3.141590465
3072	3.141592928	3.141592108

The increase in π accuracy with increasing polygon side length is seen to be very slow. Archimedes must of recognized this and therefore stopped at a polygon of 96 sides. His answer there is good to only three decimal points, namely, $\pi \approx 3.14$. At 3072 sides, we find the five decimal accurate result $\pi \approx 3.141592$. This is a lot of work required to approximate a circle by a 3072 sided regular polygon. With modern day computers it is of course possible to take the Archimedes approach to extremes. So , for example, when the number of sides equals $3(2^{100}) = 3.8029 \times 10^{30}$ we get the 61 decimal accurate result-

$$\pi=3.14159265358979323846264338327950288419716939937510582097494 4$$

For most practical purposes it is sufficient to use the five digit accurate result $\pi=3.14159$. A convenient way to remember this approximation (assuming you don't have access to a hand calculator or a good memory for numbers) is to recall the Otto Ratio $\pi \approx 355/113=3.141592$ or by making use of the German mnemonic "Drei komma Huss Verbrannt" which when translated reads 3.1415. Here the 1415 refers to the year the Czech church reformer.Huss was burned at the stake for heresy by the catholic inquisition. I first heard about this mnemonic from my now deceased friend and expert in boundary layer theory Dr. Karl Pholhausen. He remembered it from his elementary school days back in the early nineteen hundreds.

VAN CEULEN'S EXPANSION FOR PI:

Although there were many further expansions for π in the nearly 19 hundred years after Archimedes and prior to the invention of calculus, it was Ludolph van Ceulen(1540-1610) of Leiden University in the Netherlands who took the Archimedean method to its extreme. Van Ceulen was born in Germany but migrated as a young man to the Netherlands. There as a professor of mathematics he spent a good part of his adult life finding the value of π to 35 digit accuracy. He had these thirty-five digits engraved on his tombstone where it may still be seen today. In retrospect van Ceulen's efforts were somewhat in vain considering that far more accurate results were obtainable after the invention of calculus which made arctan formulas and AGM methods possible. However, in honor of his work, school children in Germany still refer to Pi as the Ludolph number..

The essence of Van Ceulen's approach was essentially identical to the Archimedes technique. He started with a square (and not a hexagon) and placed it into the annular region between an external circle of radius R and an internal circle of radius r. Then comparing areas, he obtained the first approximation-

$$2\sin(\pi/2)= 2<\pi< 2\sin(\pi/2)/\cos(\pi/4)^2 =4 \qquad ,$$

By next doubling the polygon edges to form an octagon he found-

$$4/\mathrm{sqrt}(2)=2.828..<\pi<8/(1+\mathrm{sqrt}(2))=3.311\ldots$$

Going on to 16-32-64- etc sided polygons he found the brackets on π to be-

$$2^n \sin(\frac{\pi}{2^n}) < \pi < 2^n \frac{\sin(\frac{\pi}{2^n})}{\cos(\frac{\pi}{2^{n+1}})^2}$$

for a regular 2^{n+1} sided regular polygon. Here is a short table giving bounds on π as a function of polygon side number-

No. of Sides	Lower Bound for Pi	Upper Bound for Pi
2^6	3.13	3.14
2^{11}	3.141595	3.141587
2^{21}	3.141592653585	3.141592653592
2^{41}	3.14159265358979323846264 23	3.1415926535897932384626439
2^{51}	3.14159265358979323846264338 3281	3.1415926535897932384626433 83279

So at n=50 , where the side number equals $2^{51,}$ we get π accurate to 29 digits . Van Ceulen actually went a bit further to 2^{71} sides which gave π accurate to 35 places. That he was able to accomplish this accurately without mistakes in 1596 (and before the advent of calculus and electronic computers) is truly amazing, especially for the stamina needed to repeatedly apply double angle formulas. What is clear from both Archimedes and Van Cuelen 's work is that methods based on polygon exhaustion are notoriously slow for generating good π estimates. Arctan and Algebraic-Geometric Mean Methods (AGM) are far superior(see "A History of Pi" by Petr Beckmann).

IMPROVED VERSION OF AN EXHAUSTION TECHNIQUE FOR FINDING PI:

In thinking about the Archimedes problem of using inscribed and circumscribed circles bounding n sided polygons and realizing the technique is slowly converging, it became clear to me that one should be able to use a parallel procedure not dependent on polygons but rather one an integral with rapidly oscillating integrand. Such an integral should approach zero value but be equal the difference of two large numbers , one of which contains the unknown constant π. An integral which we came up with about a decade ago is-

$$\int_{x=0}^{1} \frac{P_{2n}(x)}{a^2+x^2}dx = Const.\{N(n,a) - (\frac{1}{a})[\arctan(1/a)])M(n,a)\}$$

Here $P_{2n}(x)$ are the even Legendre polynomials which have n zeros in 0<x<1, and N(n,a) and M(n,a) are large polynomials in n. The larger n becomes the smaller will be the integral value, while at the same time the absolute values of N and M become larger and larger. Of particular interest for us here is when a=1, for then (1/a)arctan(1/a)=π/4. Working out the first five n cases for this limiting form produces-

$$\int_{x=0}^{1} \frac{P_2(x)}{x^2+1}dx = \frac{1}{2}\{3-\pi\}$$

$$\int_{x=0}^{1} \frac{P_4(x)}{x^2+1}dx = -\frac{1}{24}\{160-51\pi\}$$

$$\int_{x=0}^{1} \frac{P_6(x)}{x^2+1}dx = \frac{1}{20}\{644-205\pi\}$$

$$\int_{x=0}^{1} \frac{P_8(x)}{x^2+1}dx = -\frac{1}{1120}\{183296-58345\pi\}$$

$$\int_{x=0}^{1} \frac{P_{10}(x)}{x^2+1}dx = \frac{1}{5040}\{4317632-1374345\pi\}$$

We see from these results that N(n) and M(n) increase rapidly in size and that we expect-

$$\pi = \frac{\lim}{n \to \infty}\{\frac{-4N(n)}{M(n)}\}$$

The increase in accuracy for π with increasing n is here much faster than in the standard Archimedes approach of inscribed and circumscribed polygons. We can see this in the following table-

n	-4N/M	Decimal Place Accuracy
1	3. …	0
2	3.1…	1
4	3.1415…	4
8	3.14159265358…	11
16	3.14159265358979323846264…	23
32	3.14159265358979323846264338327950288419716939937…	48
64	3.1415926535897932384626433832795028841971693993751 058209749445923078164062862089986280348253421170…	98

So the doubling of n produces an accelerating accuracy for π. For n=32 we already exceed the Van Ceulen resuly of 35 digit accuracy ending in 288. For n=64 the value of π is given to 98 digit accuracy. Remember for the present approach one only needss to integrate integer powers of x and no roots need to be taken. With aid of the MAPLE mathematics program one has the very simple forms-

N(n)=int(quo(P(2*n,x),x^2+1,x),x=0..1):

and-

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N(n)=rem(P(2*n,x),x^2+1,x);
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To improve the convergence rate even further it is necessary to introduce values of ‘a’ greater than one. So, for instance, one could use the Machin formula-

$$\pi - 16\arctan(1/5) - 4\arctan(1/239)$$

Here(1/5) arctan(1/5)converges to accurate values of π even faster using the above technique than using (1/1)arctan(1)=π/4. An even better arctan formula to apply the present exhaustion method to apply our own formula-

$$\pi = 48\arctan(\frac{1}{38}) + 80\arctan(\frac{1}{57}) + 28\arctan(\frac{1}{239}) + 96\arctan(\frac{1}{268})$$

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