POLYNOMIAL QUOTIENT APPROXIMATIONS FOR ARCTAN(1/N) AND ARCSIN(1/N) FOR LARGE N

The functions arctan(x) and arcsin(x) have often been used to find approximations for Pi. In particular, we are thinking of the Machin formula-

\[ \pi = 16\arctan(1/5) - 4\arctan(1/239) \]

and the Newton formula-

\[ \pi = 6\arcsin(1/2) = 3 + \frac{1}{8} + \frac{9}{640} + \frac{15}{7168} + \ldots \]

Both these expressions are evaluated using a standard Taylor series expansion about \( x=0 \). These series have the form-

\[ \arctan\left(\frac{1}{N}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)N^{2n+1}}, \quad \arcsin\left(\frac{1}{N}\right) = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 (2n+1)4^n N^{2n+1}} \]

and clearly will not converge very rapidly unless \( N\gg1 \). These expansions for Pi will thus require a large number of terms in the series to produce high accuracy. For example, it will require the first 11 terms in the arctan(1/5) series in the Machin formula to achieve a 16 place accuracy in Pi. Improvements can be gotten with multiple term formulas which have been developed over the years. One of the latest is our own four term expression-

\[ \pi = 48\arctan\left[\frac{1}{38}\right] + 80\arctan\left[\frac{1}{57}\right] + 28\arctan\left[\frac{1}{239}\right] + 96\arctan\left[\frac{1}{268}\right] \]

which is seen to contain only large values of N. Indeed, to get an equivalent 16 place accuracy for Pi from this last formula will require just the first five terms in the series for arctan(1/38), with even fewer terms required for the three other arctan terms in the formula. It is possible to develop an infinite number of other arctan formulas with large N, although this seems not to have been considered in the past.

We have spent some time in trying to find a method to accelerate the convergence rate of both the arctan and arcsin series expansions given above but have had only limited success and in no way can approach in utility the presently used AGM-elliptic integral methods for finding ever more accurate values of Pi. Nevertheless, we have come up with some polynomial quotient expressions for arctan and arcsin which may be of interest for quickly approximating values of these functions for large N. We present here some of these developments.
Start with the basic definitions-

\[
\arctan\left(\frac{1}{N}\right) = \int_{0}^{1/N} \frac{dx}{1 + x^2}, \quad \arcsin\left(\frac{1}{N}\right) = \int_{0}^{1/N} \frac{dx}{\sqrt{1 - x^2}}
\]

and then use the substitution \(Nx = \tanh(z)\). This leads after a short manipulation to the infinite range integrals-

\[
\arctan\left(\frac{1}{N}\right) = N\varepsilon \int_{0}^{\infty} \frac{dz}{cosh(z)^2 - \varepsilon}, \quad \arcsin\left(\frac{1}{N}\right) = \sqrt{\beta} \int_{0}^{\infty} \frac{dz}{cosh(z)\sqrt{cosh(z)^2 + \beta}}
\]

where \(\varepsilon = 1/(N^2 + 1)\) and \(\beta = 1/(N^2 - 1)\) are small parameters. Since \(N\) is large, we can expand the integrand in a Taylor series involving \(\varepsilon\) and \(\beta\) and then integrate the resultant integrals analytically. For the arctan function one finds-

\[
\arctan\left[\frac{1}{N}\right] = N\varepsilon \int_{0}^{\infty} \frac{dz}{cosh(z)^2} \left\{1 + \frac{\varepsilon}{cosh(z)^2} + \frac{\varepsilon^2}{cosh(z)^4} + \cdots\right\}
\]

\[
= N\varepsilon \sum_{n=0}^{\infty} \frac{4^n n^2 \varepsilon^n}{(2n + 1)!} = \varepsilon N \left\{1 + \frac{2}{3} \varepsilon + \frac{8}{15} \varepsilon^2 + \cdots\right\}
\]

This series is one already found by Euler and converges at about the same rate as the well known result given earlier. An interesting observation is that it produces the identity-

\[
\frac{\pi}{2} = 1 + \frac{1}{3!} + \frac{2^2}{5!} + \frac{2^3}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{2^n (n!)^2}{(2n + 1)!}
\]

The series found for arcsin is-

\[
\arcsin\left(\frac{1}{N}\right) = \sqrt{\beta} \int_{0}^{\infty} \frac{dz}{cosh(z)^2} \left\{1 - \frac{1.\beta}{2cosh(z)^2} + \frac{1.3\beta^2}{2^22!cosh(z)^4} - \cdots\right\}
\]

\[
= \sqrt{\beta} \sum_{n=0}^{\infty} \frac{(-1)^n \beta^n}{(2n + 1)} = \sqrt{\beta} \left\{1 - \frac{\beta}{3} + \frac{\beta^2}{5} - \frac{\beta^3}{7} + \cdots\right\}
\]

This last series has a somewhat simpler form than the standard series for arcsin given earlier, but its convergence rate is no better.
In further looking at the infinite range integral representations above, one notices that they are functions of both the variables \( z \) and \( N \). This suggests applying the Leibnitz rule to \( N \) to generate some additional equalities of possible interest. One differentiation of the above arctan integral yields:

\[
\arctan\left(\frac{1}{N}\right) = \frac{N}{N^2 - 1} - \frac{2N^3}{(N^2 - 1)(N^2 + 1)^2} \int_0^\infty \frac{dz}{\cosh(z)^2 - \frac{1}{(N^2 + 1)^2}}
\]

and differentiating a second time produces:

\[
\arctan\left(\frac{1}{N}\right) = \frac{3N(N^2 - 1)}{(3N^4 - 2N^2 + 3)} + \frac{8N^5}{(N^2 + 1)^3 (3N^4 - 2N^2 + 3)} \int_0^\infty \frac{dz}{\cosh(z)^2 - \frac{1}{(N^2 + 1)^3}}
\]

What will be noticed is that the term involving the integral in the above results become progressively smaller relative to the first term. In the last result the first term is of order \( 1/N \) and the second term goes as \( 1/N^5 \), so that for larger \( N \) the first polynomial quotient gives a reasonable estimate without needing to integrate. The differentiation can be easily automated. Calling these polynomial quotients \( Q_1, Q_2, Q_3, \) etc. and evaluating for \( N=38 \) we have the following:

\[
Q_1 = \frac{38}{38^2 - 1} = 0.0263340263
\]

\[
Q_2 = \frac{3 \times 38 \times (38^2 - 1)}{(3 \times 38^4 - 2 \times 38^2 + 3)} = 0.0263096993
\]

\[
Q_3 = \frac{38 \times (15 \times 38^4 - 14 \times 38^2 + 15)}{(15 \times 38^6 - 9 \times 38^4 + 9 \times 38^2 - 15)} = 0.0263097172
\]

which compares to:

\[
\arctan(1/38) = 0.0263097172
\]

and thus shows a ten place accuracy for the quotient obtained by just three differentiations. Even more accurate quotients are obtainable by increasing the number of differentiations, but this, unfortunately leads to very large polynomials and thus has to be carried out by computer. The required iteration program is very simple, namely

\[
F_1 = \frac{1}{\varepsilon N} \arctan\left(\frac{1}{N}\right) , \quad F_{n+1} = -\frac{1}{2n\varepsilon^2} \frac{dF_n}{dn}
\]

with the nth quotient given by:

\[
Q_n = -\frac{A_n}{B_n} \quad where \quad F_n = A_n + B_n \arctan\left(\frac{1}{N}\right)
\]
Note that the exact value for \( \arctan(1/N) \) is given by \( \lim_{n \to \infty} [Q_n] \) and the quotient \( Q_n \) will approach this limit with increasing rapidity the larger \( N \) becomes. We note that these quotients are really just Pade approximates which will match the standard infinite series expansion up to the \( n \)th term. Thus, for example,

\[
Q_3 = \frac{N(15N^4 - 14N^2 + 15)}{(15N^6 - 9N^4 + 9N^2 - 15)} = \frac{1}{N} - \frac{1}{3N^3} + \frac{1}{5N^5} + \frac{33}{25N^7} + \cdots
\]

A similar Leibnitz rule application to the above arcsin integral yields upon one differentiation-

\[
\arcsin\left(\frac{1}{N}\right) = \frac{1}{N^2 \sqrt{\beta}} + \beta^{3/2} \int_0^\infty \frac{dz}{\{\cosh(z)[\cosh(z) + 1]\}^{3/2}}
\]

and a second differentiation yields the quotient approximation-

\[
\arcsin\left(\frac{1}{N}\right) \approx \sqrt{\beta} \frac{(3N^8 - 7N^6 + 3N^4 + 3N^2 - 2)}{N^4 (3N^4 - 6N^2 + 3)}
\]

where we recall that \( \beta = 1/(N^2-1) \) so that \( \arcsin(1/N) \) goes as \( 1/N \) for extremely large \( N \). We find that this last quotient gives the eight place accurate result at \( N=38 \) of \( \arcsin(1/38)=0.02631882 \). The iteration for the arcsin approximations is given by-

\[
G_1 = \frac{1}{\sqrt{\beta}} \arcsin\left(\frac{1}{N}\right) \quad \text{followed by} \quad G_{n+1} = \frac{1}{(2n-1)\beta^2 N} \frac{dG_n}{dN}
\]

and is a procedure easily implemented by a one line computer program using a canned program such as MAPLE.

Overall one can conclude that the quotient expressions found above are just \( n \) term accurate representations of the infinite series expansions given earlier and so do not really improve upon these series as far as the number of iterations required to reach \( n \) term accuracy. The main thing to remember is that the larger \( N \) becomes the more rapidly things will converge and the more useful these quotients and arctan formulas for Pi become.

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