The famous Swiss mathematician Jacob Bernoulli (1655-1705) was the first person to succeed in finding the sum of the first m integers each taken to an integer power p. He found this sum by induction based on the following set of known results:

\[
\sum_{n=1}^{m} n^1 = \frac{m^2}{2} + \frac{m}{2}
\]

\[
\sum_{n=1}^{m} n^2 = \frac{m^3}{3} + \frac{m^2}{2} + \frac{m}{6}
\]

\[
\sum_{n=1}^{m} n^3 = \frac{m^4}{4} + \frac{m^3}{2} + \frac{m^2}{4}
\]

\[
\sum_{n=1}^{m} n^4 = \frac{m^5}{5} + \frac{m^4}{2} + \frac{m^3}{3} - \frac{m}{30}
\]

\[
\sum_{n=1}^{m} n^5 = \frac{m^6}{6} + \frac{m^5}{2} + \frac{5m^4}{12} - \frac{m^2}{12}
\]

\[
\sum_{n=1}^{m} n^6 = \frac{m^7}{7} + \frac{m^6}{2} + \frac{m^5}{2} - \frac{m^3}{6} + \frac{m}{42}
\]

\[
\sum_{n=1}^{m} n^7 = \frac{m^8}{8} + \frac{m^7}{2} + \frac{7m^6}{12} - \frac{7m^4}{24} + \frac{m^2}{12}
\]

He saw at once from this set that-

\[
\sum_{n=1}^{m} n^p = \frac{m^{p+1}}{p+1} + \frac{m^p}{2} + \frac{pm^{p-1}}{12} - \frac{(p)(p-2)m^{p-3}}{120} + \ldots
\]

\[
= \frac{1}{(p+1)} \left\{ B_0 m^{p+1} - B_1 \frac{(p+1)}{1} m^p + B_2 \frac{(p+1)}{2} m^{p-1} - \ldots \right\}
\]

where \( B_0 = 1, B_1 = -1/2, \) and \( B_2 = 1/6 \) are constants now known as the Bernoulli Numbers. These constants were later shown to be given by the generating formula-

\[
\frac{t}{\exp(t) - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}
\]

A special case for \( t=1 \) states that –
\[ \sum_{n=0}^{\infty} \frac{B_n}{n!} = \frac{1}{\exp(1) - 1} = 0.581976... \]

On expanding both sides of the generating function one gets-

\[ 1 - \frac{1}{2}t + \frac{1}{6(2!)} t^2 - \frac{1}{30(4!)} t^4 + \frac{1}{42(6!)} t^6 - \frac{1}{30(8!)} t^8 + .. = B_0 + B_1 \frac{t^1}{1!} + B_2 \frac{t^2}{2!} - B_3 \frac{t^3}{3!} + B_4 \frac{t^4}{4!} - B_5 \frac{t^5}{5!} + B_6 \frac{t^6}{6!} - \frac{7}{7!} + B_8 \frac{1}{8!} + ... \]

Looking at the various powers of \( t \) we find the first few Bernoulli Numbers to be-

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_n )</td>
<td>1</td>
<td>-1/2</td>
<td>1/6</td>
<td>0</td>
<td>-1/30</td>
<td>0</td>
<td>1/42</td>
<td>0</td>
<td>-1/30</td>
</tr>
</tbody>
</table>

You will notice that the odd \( n \) Bernoulli Numbers vanish except for \( n=1 \). This suggests we also look at the slightly altered expansion-

\[ \frac{t}{\exp(t) - 1} - \frac{1}{2} + \frac{t^2}{12} \left\{ 1 - \frac{t^2}{60} + \frac{t^4}{2520} - \frac{t^6}{100800} + \frac{t^8}{3991680} - .. \right\} \]

On rewriting this equation by letting \( k = n - 2 \) we get the new generating formula-

\[ \left\{ 1 - \frac{t^2}{60} + \frac{t^4}{2520} - \frac{t^6}{100800} + \frac{t^8}{3991680} - .. \right\} = 12 \sum_{k=0}^{\infty} B_{k+2} \frac{t^k}{(k+2)!} \]

We see here that the infinite series is alternating and contains only even powers of \( t \). Hence all odd Bernoulli Numbers beyond \( B_1 \) must be zero. Note that at \( k = 6 \) we have-

\[ B_8 = \frac{-8!}{12(100800)} = \frac{-1}{30} \]

Another series which generates Bernoulli Numbers is gotten via the expansion-

\[ \frac{t}{1 - \exp(-t)} - \frac{1}{2} + \frac{t^2}{12} \left\{ 1 - \frac{t^2}{42} + \frac{t^4}{1680} - \frac{t^6}{66528} + \right\} \]
This produces-

\[-720 \sum_{n=0}^{\infty} B_{k+4} \frac{t^k}{(k+4)!} = \left[1 - \frac{t^2}{42} + \frac{t^4}{1680} - \frac{t^6}{66528} + \ldots\right]\]

So for \(k=6\) we find that-

\[B_{10} = \frac{10!}{720(66528)} = \frac{5}{66}\]

In the actual calculation of Bernoulli Numbers by electronic computer one uses the recurrence relation-

\[B_n = -\frac{1}{(n+1)} \sum_{k=0}^{n-1} \frac{(n+1)!}{k!(n+1-k)!} B_k\]

The well known Victorean female mathematician lady Ada Lovelace came up with this recurrence relation to calculate Bernoulli Numbers. She was the daughter of British poet Lord Byron and spent many years of her short life working alongside Charles Babbage, the inventor of the first mechanical calculating machine. The above recurrence relation written in MAPLE language reads-

\[\text{bernoulli}(n)=-1/(n+1)*\text{sum}(\text{bernoulli}(k)^*(n+1)!/(k!*(n+1-k)!),k=0..n-1);\]

for \(B_{20}\) it produces-

\[B_{20} = -\frac{174611}{330}\]

We point out that Bernoulli Numbers appear in many other areas in the mathematics repertoire. For instance, the infinite series expansion for \(\tan(z)\) is-

\[\tan(z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n} (2^{2n} - 1) B_{2n} z^{2n-1}}{(2n)!} = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + \ldots\]

In addition one can find additional identities involving the Bernoulli Numbers in the Abramowitz and Stegun “Handbook of Mathematical Functions”. One of the more interesting of these is-
\[ \zeta(2n) = \sum_{k=0}^{\infty} \frac{1}{k^{2n}} = \pm \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \]

Here the plus sign refers to odd integer \( n \) while the minus sign applies for even integer \( n \). Since one knows the value of the zeta function for many \( \zeta(2n) \), the value for \( B_{2n} \) will follow directly. Take \( n=7 \). There we get-

\[
B_{14} = \left( \frac{2\pi^{14}}{18243225} \right) \left( \frac{2(14!)}{(2\pi)^{14}} \right) = \frac{7}{6}
\]

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