HOW ARE THE BINOMIAL EXPANSION, THE PASCAL TRIANGLE, COIN FLIPPING AND THE GAUSSIAN ALL RELATED TO EACH OTHER?

Most STM majors will become familiar in their undergraduate days with advanced algebra, including the binomial expansion of $(x+y)^n$, some statistics where probability , variance, standard deviation and the Gaussian are discussed, plus courses in differential and finite difference equations, and computer programming. What is lacking in many of these courses is that nowhere is there an attempt made to tie many of the fundamental concepts taught in them to a simple common base. For example, many science, technology, and mathematics majors are at a loss upon graduating to explain how the binomial theorem, the Pascal Triangle, coin tossing , Heat Flow in a Solid, and the Universal Density Distribution of Gauss are all part of the same package. It is our purpose here to discuss how one can, starting with the simple binomial expansion, explain concepts such as the Pascal Triangle, Probability in Coin Flipping, Temperature expansion from a Hot Spot, and the origin of the well known Gaussian.

We begin with an expansion of the function $F=(x+y)^n$ for any positive integer n. Carrying out such an expansion starting with n=0 one has-

$$
\begin{aligned}
(x+y)^0 &= 1\\
(x+y)^1 &= x+y\\
(x+y)^2 &= x^2+2xy+y^2\\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\
(x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4
\end{aligned}
$$

On generalizing , these results can be written in compact form as-

$$(x+y)^n = \sum_{k=0}^{n} C[n,k]x^{n-k}y^k$$

, where C[n,k] is the Binomial Coefficient defined as-

$$C[n,k] = \frac{n!}{k!(n-k)!}$$

This coefficient has numerous properties including that-

C[n,0]=C[n,n]=1 , C[n,k-1]+C[n.k]=C[n+1,k] , and $\sum_{k=0}^{n} C[n,k] = 2^n$

Also when n is even, the coefficient is symmetric about k=n/2 reaching its peak value at that point. A convenient way to write out the Binomial Coefficient is via then following Pascal Triangle-

1	n=0	sum
1 1	1	2
1 2 1	2	4
1 3 3 1	3	8
1 4 6 4 1	4	16
1 5 10 10 5 1	5	32
1 6 15 20 15 6 1	6	64

Here the third row C[3,k] reads 1-3-3-1 and the sixths row C[6,k]=1-6-15-20-15-6-1

Before Newton and his introduction of calculus, the Pascal Triangle was the standard way to determine the coefficients in a binomial expansion. With modern computers it is of course much simpler to just evaluate the factorial quotient defining C[n,k] for any n. Note that the elements in each row add up to precisely 2^n. The central column contains an element only when n is even. The value of this element is $n!/[(n/2)!]^2$. So for n=30 we get the peak value to be $30!/(15!)^2=155117520$.

One can also normalize C[n,k] by dividing each term in a row by 2^n. This produces the modified Pascal Triangle-

1

1/2 1/2

1/4 1/2 1/4

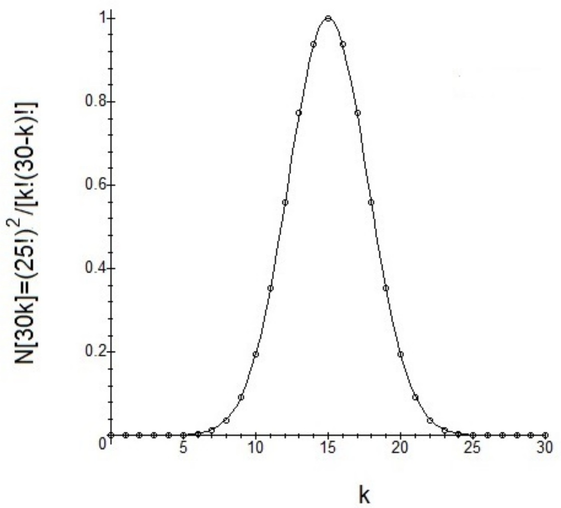
1/8 3/8 3/8 1/8

1/16 1/4 3/8 1/4 1/16

1/32 5/32 5/16 5/16 5/32 1/32

Now each row adds up to precisely one. For the even integer n=30 one gets the following symmetric pattern for the normalized 30^{th} row C[30,k]-

NORMALIZED BINOMIAL COEFFICIENT N[30,k] VERSUS
THE GAUSSIAN G=EXP[-(15-k)^2/15.2301]



The circles represent the normalized 31 values of C[30,k] from k=0 through k=30. The element magnitudes follow almost exactly the form of the superimposed Gaussian-

$$G=\exp\{-(k-15)^2/15.2301\}$$

Indeed , as n goes to infinity the elements C[n,k] for fixed n will coincide with the Normal(Gaussian) Distribution Density-

$$G[x]=\frac{1}{\sqrt{2\pi\sigma^2}}\exp-\left[\frac{(x-\mu)^2}{2\sigma^2}\right]$$

as one encounters in statistics. This curve when integrated over the infinite range $-\infty<x<\infty$ is always equal to unity regardless of the mean value μ or the standard deviation σ. The inflection points of this Gaussian lie at $x=\mu\pm\sigma$. The following ten mark German note was printed in honor of Carl F. Gauss (1777-1855) showing his famous curve-

Going back to the normalized Pascal triangle given above, one realizes that it can be thought of as a probability diagram describing the flipping of a single coin. The first flip shows that heads(H) and tails(T) have the same probability of P=1/2 . The second row represents the probability results for two successive flips. The possible outcomes are HH-HT-TH-TT with two heads or two tails having the probability of P=1/4 each, while the head-tail combination yields the higher probability of P=1/2. Next reading the elements in the third row we get the probability of three heads in a row or three tails in a row being P=1/8 with the remaining combinations of HHT and HTT yielding P=3/8 each. The largest probability after 4 flips will be HHTT where P=3/8. Getting four heads in a row has the lower probability of P=1/16.

Next we look at the classic problem of one dimensional heat conduction in an infinitely long bar insulated on its sides with an initial temperature of zero except at x=0 where T is infinite. The heat conduction process in this bar is governed by the partial differential equation and initial and boundary conditions given by-

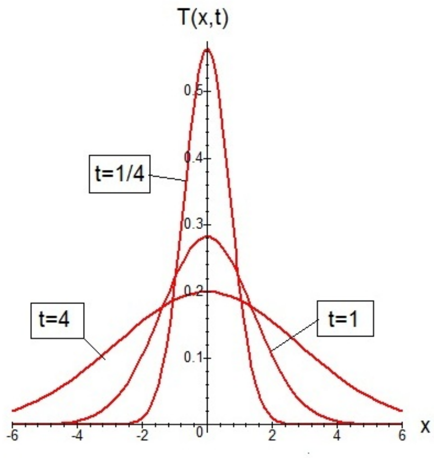
$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \quad subject \quad to \quad T(\pm\infty, t) = 0 \quad and \quad T(x,0) = \delta$$

Here the time t is normalized by the thermal diffusivity and δ is the Dirac delta function used to simulate the hot spot. On taking a Fourier Transform of this equation with respect to x, solving the resultant first order differential equation, and then inverting , produces the result-

$$T(x,t) = \frac{1}{2\sqrt{\pi t}} \exp(-x^2 / 4t)$$

A graph for the times 1/4, 1, and 4 follows –

TEMPERATURE SPREADING OUT OF A HOT SPOT



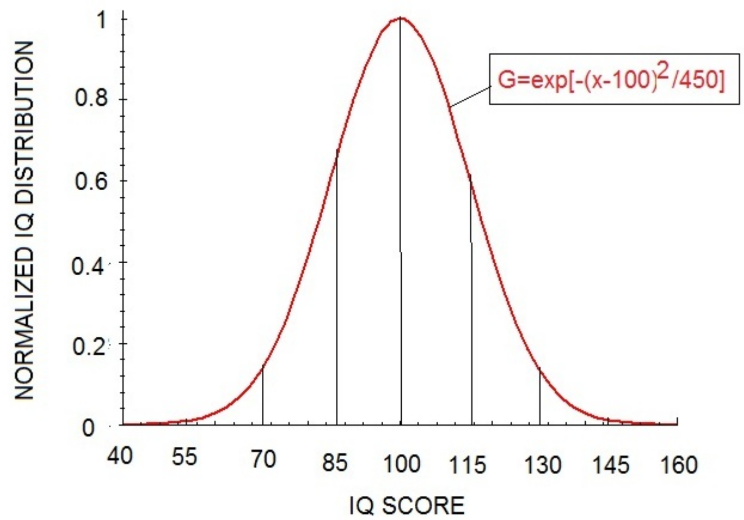
T(x,t)=1/[2 sqrt(Pi t)]exp(-x^2/4t)

We see here a typical Gaussian shape where the area underneath all three curves is unity. The Gaussian widens from zero width at t=0, where it represents a Dirac Delta function, to progressively wider symmetric Gaussians as t is increased. On integrating T(x,t) over the range $-\infty<x<\infty$ for fixed t one always gets unity, as is to be expected because no heat is lost through the sidewalls of the bar.

Finally let us look some more at the Gaussian, this time in its role in IQ determination. IQ tests were first developed in the early part of the 19 hundreds to test the mental acuity of recruits for the military services and were later extended to students in public schools and colleges and also for job applicants. In more recent years considerable controversy has arisen with regard to fairness of these tests for minority groups and those without proper language skills. Nevertheless they are still used extensively and correlate well with other measures of ability such as SAT scores.

The idea behind IQ tests is that in statistics, when the sample gets large, there will be a Gaussian spread about the average performance. It was noted that a standard deviation of σ=15 points from the mean at μ=100 gave a reasonable measure of test performance. With these values for μ and σ, one gets the following Gaussian curve when normalized to unity at x=μ=100-

IQ DISTRIBUTION OF THE US POPULATION USING A STANDARD DEVIATION OF FIFTEEN



If one multiplies the normalized distribution above by 1/sqrt(2π σ²), the total area under the Gaussian becomes one. An IQ of 140 or above means that the area remaining will be just-

$$1-\int_{-\infty}^{140}\frac{1}{\sqrt{450\pi}}\exp\{-(x-100)^2/450\}=(\frac{1}{2})[1-erf\,(4\sqrt{2}/3]=0.00383...$$

This means that a person with an IQ of 140 will be among the top 4 scorers taking the test out of 1000 individuals. Many years ago while my daughter was still in middle school she became a member of MENSA. Mensa restricts its membership to individuals with an IQ higher than 98% of the population. Recently the internet has given IQ estimates for our presidents going back to George Washington. The stated scores of 150 and up are highly suspicious. I certainly would not place Truman, Eisenhower, Johnson, Reagan , Clinton, Ford, the Bushes, Obama, and especially Trump anywhere near the stated scores . Our most brilliant president in my opinion was Thomas Jefferson.

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