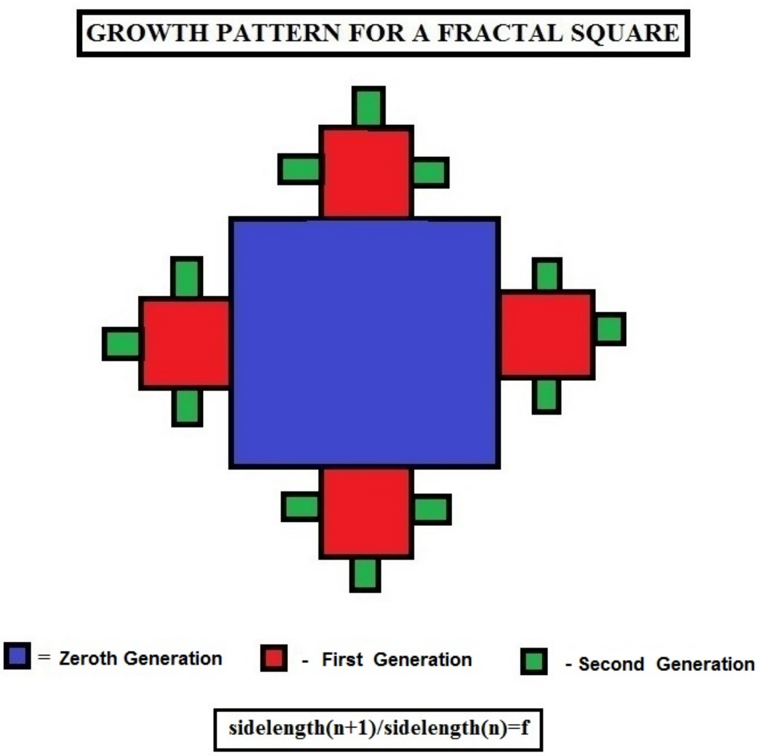
THE BLACK SNOWFLAKE AND RELATED CELLUAR AUTOMATA

A cellular automaton is an intricate 2D figure generated by a simple starting figure and then have it propagate with different magnification through subsequent generations. One of the simplest of these starts with a square of side-length 1 as the zeroth generation. One adds to this four smaller squares centered on the sides of the first square. The side-length of the smaller squares are equal to f, where $0<f<1$. This makes up the first generation. It is followed by nine even smaller centered squares of side-length f^2 each. This constitutes the second generation. Through the second generation these manipulations produce the following fractal automaton-



Since the shapes added by each generation remain squares we can also refer to such figures as fractals. If the magnification factor remains smaller than f=sqrt(2)-1=0.41421,, all subsequent generations are allowed to exist without colliding with earlier generations. This fact is established by looking at the shortest distance between the first and third generation. However, since the squares in each generation become smaller in area by a factor of f^2, the total area of this cellular automaton will approach a finite value. Here is a list of the total area of all squares through generation n-

$$A(0) = 1 \quad (blue)$$
$$A(1) = 4f^2 \quad (red)$$
$$A(2) = 3 \cdot 4f^4 \quad (green)$$

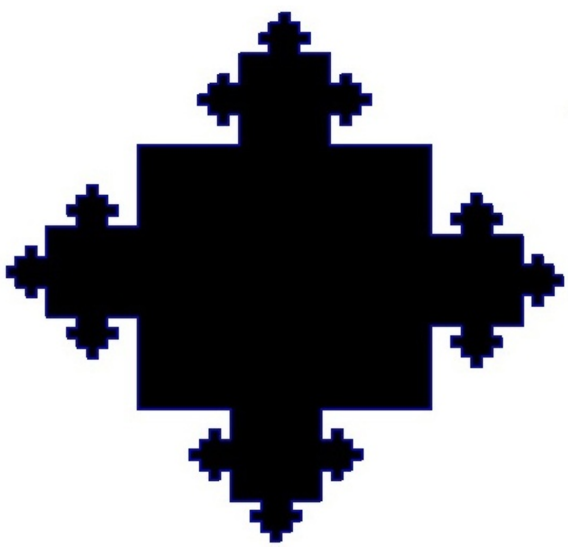
Adding up these areas through the nth generation yields the total area-

$$T(n) = 1 + 4f^2 + 4 \cdot 3f^4 + 4 \cdot 3^2 f^6 + ... + 4 \cdot 3^{n-1} f^{2n}$$
$$= [1 + 4f^2 \sum_{k=0}^{n-1} (3f^2)^k]$$

From this result we have that

$$T(\infty) = \frac{(1+f^2)}{(1-3f^2)}$$

An aesthetically pleasing form we call the Black Snowflake occurs for f=1/3. It has a total finite area of T(∞)=5/3 which is just slightly shy of 2. Here is this snowflake as it looks through the third generation-



Note that the pattern is bounded by a 45 deg rotated square of side-length sqrt(2). This means that for f=1/3 the void fraction left inside the bounding square (after n is allowed to go to infinity) is-

$$Void\ Fraction = 1 - \frac{5}{6} = 0.1667..$$

There is an important point to note in the above calculations. It is that it will not be possible to construct a complete snowflake going out to n equals infinity unless f is small enough. To find this critical value for f we note from the above figures that-

$$\frac{f}{2}(1-f)^2 \geq \{1-(1+f+f^2)(1-f)\}$$

That is-

$$0 \geq f^2 + 2f - 1$$

which means that-

f<sqrt(2)-1=0.41421…

From this result we see that f=1/3 allows for a snowflake with an infinite number of generations but f=1/2 will not. Ineed f=1/2 produces a collision between generations after reaching n=3.

We next examine the perimeter of the snowflake pattern for f<sqrt(2)-1. The simplest way to calculate this perimeter is to write down the perimeter of each of the generations and then add. It is important to remember that when the nth generation perimeter is counted one must subtract from it the contact lengths of the n+1 and n-1 generation. Thus we have –

p(0)=4(1-f)

p(1)=4x3f(1-f)

p(2)=4x3^2f^2(1-f)

Adding these values together produces the total perimeter-

$$S(n) = 4(1-f)\sum_{k=0}^{n}(3f)^k$$

Here is a table giving the area and perimeters up through the 4^{th} generation using the magnification factor of f =1/3-

Max.Generation Included, n	Total Area, T	Total Perimeter, S	S/T Ratio
0	1	8/3	2.6667
1	13/9	16/3	3.6923
2	43/27	24/3	5.0232

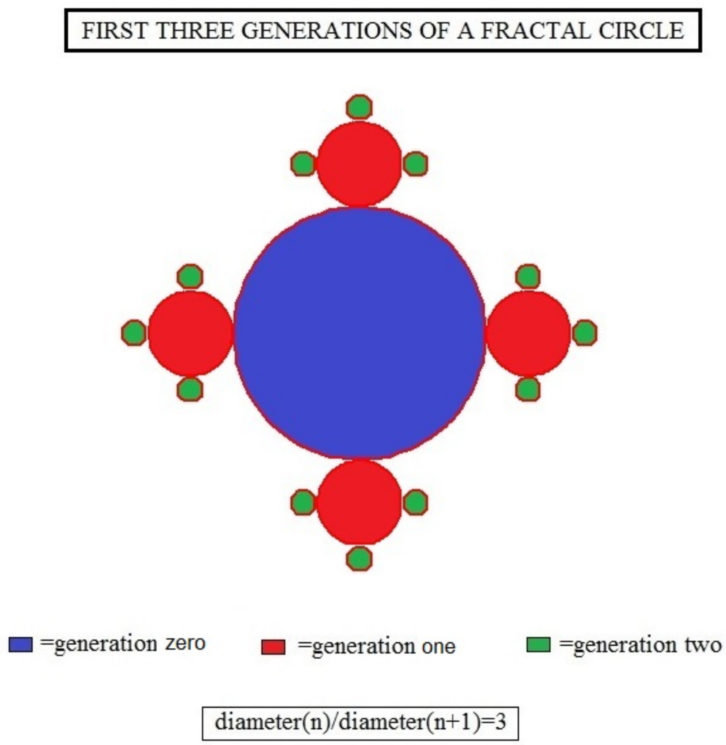
3	133/81	32/3	6.6976
4	403/243	40/3	8.0397

From the table we see two things of importance. The first shows that for f=1/3 we have the perimeter through the nth generation equal to-

$$S(n) = (n+1)(\frac{8}{3})$$

and that the perimeter becomes infinite as n goes to infinity while T(n) always retrains at a finite value below two for the f=1/3 case. This is the typical behavior expected of a fractal.

The above snowflake pattern can be used as the template for numerous other shapes which just fit into the zeroth generation square. For example, one can start with a circle of radius 1/2 which fits inside a square of sides 1. Adding next the first generation of four circles of radius f/2 each and followed by a collection of 12 still smaller circles of radius $f^2/2$ each, produces the following fractal of circles when the radius ratio is f=1/3-



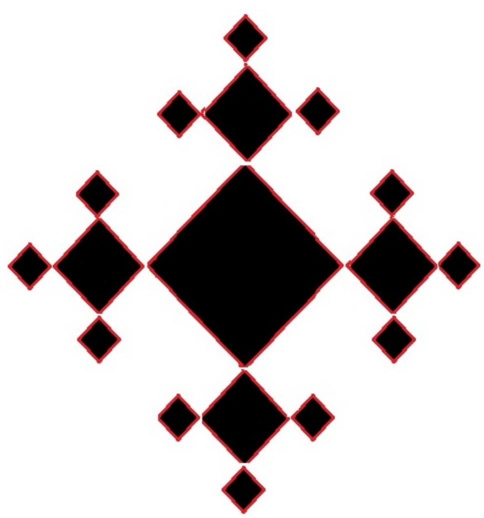
In this case there is no worry about collisions between generations since 1/3<[sqrt(2)-1]=0.41421…   That is, no circle will get outside of the Black

Snowflake template square. To determine the total area of these circles one need only use the fact that T(n) for the Black Snowflake and the total area of the circles through generation n differ from each other by a factor of $\pi/4$ which represents the area of the circle and circumscribed square for the zeroth generation.
When the figure inside the zeroth generation square just touches its cardinal points as it does for the above circle case , then a direct calculation for perimeter is straight forward. For the case of the 17 circles above, we get a total perimeter of-

$$S(2) = \pi\left\{1 + \frac{4}{3} + \frac{12}{9}\right\} = \pi\left(\frac{11}{3}\right)$$

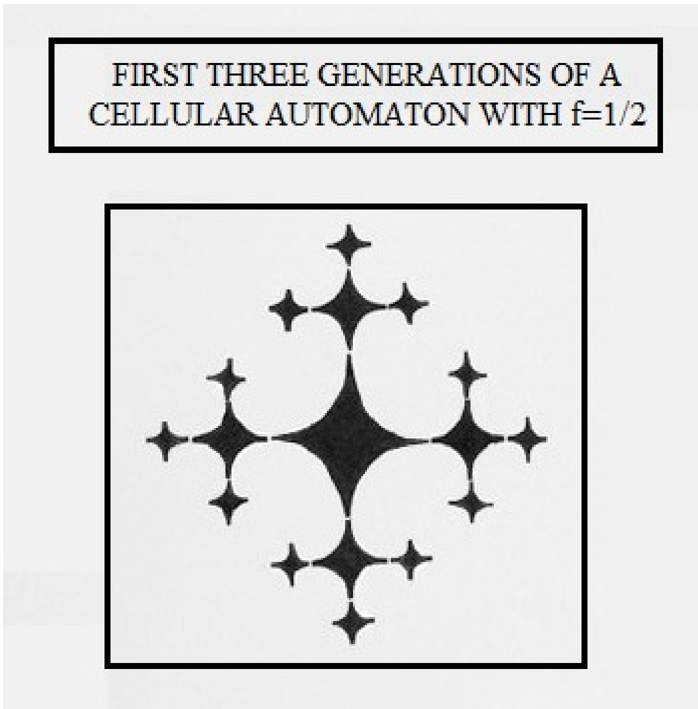
There are numerous other figures which may be used as a starting point for the zeroth generation. For example a diamond shape pattern whose corners just touch the middle of the sides of a first generation square of side-length 1, produces the following pattern when f=1/2-

FIRST THREE GENERATIONS OF A DIAMOND PATTERN USING THE BLACK SNOWFLAKE TEMPLET WITH f=1/2



Note here we will here find an overlap between generations since f=1/2>sqrt(2)-1. It means that the generation growth will stop after only a few generations.

As already mentioned there are numerous other configurations which can serve as a starting point for the development of a cellular automaton. Some of the resultant patterns can lead to esthetically pleasing forms which can serve as an inspiration for artists. We show here one such pattern produced by using a zeroth generation figure resulting from the area left over by scalloping a square with four circles of radius 1/2 whose centers lie at the corners of the square. Here is a paper collage of the resultant figure when taken through two generations-



Again, since we have taken the magnification factor as f=1/2, higher generations will run into each other. Only if we kept f to less than sqrt(2)-1 would the automaton be able to propagate through an infinite number of generations.

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