CALCULUS IN A NUTSHELL

INTRODUCTION:

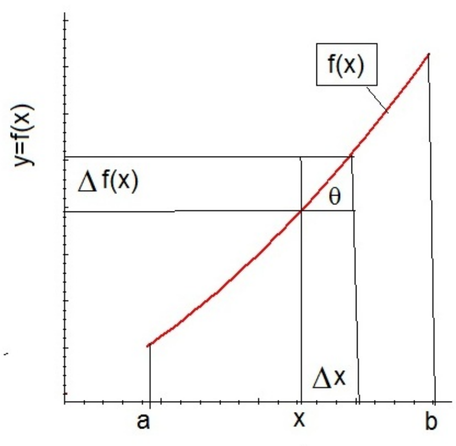
Students are usually introduced to basic calculus in twelfth grade in high school or the first year of college. The course is typically stretched out over one year and involves over stuffed books reaching lengths of five hundred pages or more and often taught by teachers poorly prepared for handling of the topic. It is our purpose here to show that calculus is a very simple and straight forward form of mathematical analysis dealing predominantly with continuous functions.

A standard calculus course consists of two parts both of which deal with continuous functions f(x) defined over an interval $a<x<b$. The first is termed Differential Calculus. It is strictly concerned with the slope (derivative) of a function. The second part deals with the area underneath a function f(x) and the x axis and forms the subject of Integral Calculus. Both Isaac Newton(1643-1727) and Gottfried Leibnitz(1646-1716) are credited with the invention of calculus.

DIFFERENTIAL CALCULUS:

Here we deal with a continuous function f(x) extending from x=a to x=b whose slope we wish to measure at a point x. we have the picture-

DEFINITION GRAPH FOR THE DERIVATIVE OF A FUNCTION



$$\text{Slope}=\tan(\theta)=\lim_{\Delta x \Rightarrow 0}\left[\frac{\Delta f(x)}{\Delta x}\right]$$

If we take a neighboring point at $x+\Delta x$ and let Δx approach zero we get the slope of f(x) to be-

$$\tan(\theta) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

In calculus we call this ratio the derivative of f(x) at x. A common way to express this derivative is as f(x)’ or df(x)/dx. We also can construct a second derivative as-

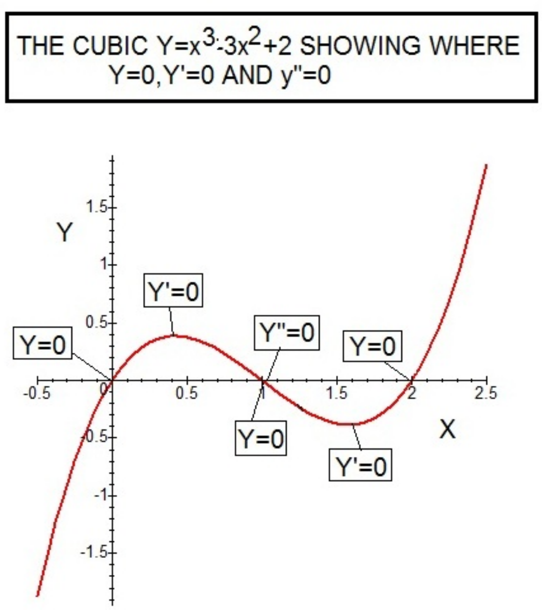
$$f(x)'' = \frac{d^2 f(x)}{dx^2} = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}$$

, again requiring Δx to approach zero. Even higher derivatives can be generated by a repeated application of the last procedure.

Let us work out a few derivatives of different functions. Consider first the power function $f(x)=x^n$. Here f(x)’ equals-

$$\frac{d(x^n)}{dx} = \frac{x^n + nx^{n-1}\Delta x - x^n}{\Delta x} = nx^{n-1}$$

We used the first two terms of a binomial expansion to get this result.
The second derivative reads $n(n-1)x^{n-2}$ and the third derivative is $n(n-1)(n-2)x^{n-3}$. The vanishing of the first derivative at x=c implies the function f(c) has zero slope at that point. Thus the cubic equation $f(x)=x^3-3x^2+2x$ has zero slope at x=1±1/sqrt(3). Its second derivative f(x)” vanishes at x=1. This is known as an inflection point. Inflection points occur whenever there is a change in the curvature of f(x). Here is a graph of this cubic-



Another function whose derivative we want to take a look at is $f(x)=a^x$, where 'a' is a constant. There we have-

$$\frac{d(a^x)}{dx}=a^x\left\{\frac{a^{\Delta x}-1}{\Delta x}\right\}$$

This is an interesting result since it shows that there is just one value of 'a' for which the term in the curly bracket equals exactly one so that the function equals its own derivative. This occurs only for the irrational number-

$$e=\sum_{n=0}^{\infty}\frac{1}{n!}=2.718281828459045....$$

which is one of the best known mathematical constants appearing in the literature. More generally one has that the term in the above curly bracket as Δx goes to zero is ln(a) , where ln refers to the natural logarithm which can be looked up on one's hand calculator. So if a=2 we find that $d(2^x)/dx= 2^x \ln(2)$.

To get the derivative of the sine function we write –

$$\frac{d[\sin(x)]}{dx}=\frac{\sin(x+\Delta x)-\sin(x)}{\Delta x}=\cos(x)$$

The quotient involving dx is an indication that the limiting form of Δx has already been taken. Note the sign change for d[cos(x)]/dx=-sin(x).

The derivative of a product is gotten as follows-

$$\frac{d[f(x)g(x)]}{dx}=f(x)\frac{d[g(x)]}{dx}+g(x)\frac{d[f(x)]}{dx}$$

while that for the quotient goes as-

$$\frac{d[f(x)/g(x)]}{dx}=\frac{-f(x)g(x)'+g(x)f(x)'}{[g(x)]^2}$$

In terms of derivatives one can expand any continuous functions with finite derivatives as a Taylor series which reads-

$$f(x+\Delta x)=f(x)+f(x)'\Delta x+f(x)''(\Delta x)^2/2!+...$$

This allows us to say that –

$$\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad , \quad \cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \quad and \quad \exp(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

INTEGRAL CALCULUS:

This second part of any calculus course involves finding the area underneath a curve f(x). It is referred to as integral calculus. To find the area between the curve f(x) and the x axis in the range a<x<b we break the area up into N=(b-a)/Δx small sub-areas of width Δx and height f(x_n). This allows us to write-

$$Area=\sum_{n=1}^{N} f(x_n)\Delta x \rightarrow \int_{x=a}^{x=b} f(x)dx$$

as Δx goes to zero. Now if we let f(x)=d[g(x)]/dx we get that the area just equals g(b)-g(a). Relaxing the end conditions we arrive at what is known as the Fundamental Theorem of Calculus, namely,-

$$g(x)=\int f(x)dx \quad where \quad dg(x)/dx=f(x)$$

Using this last result allows us to state that the area underneath the parabola f(x)=x^2 extending from x=0 to x=1 is 1/3. This is expressed mathematically as –

$$\text{Area=}\int_{x=0}^{1} x^2 dx=[\frac{x^3}{3}]_0^1=\frac{1}{3}$$

Here is a short integral table relating g(x) to f(x)-

g(x)	f(x)
x^{n+1}/(n+1)	x^n
exp(ax)	a exp(ax)
sin(ax)/a	cos(ax)
cos(ax)/a	-sin(ax)
tan(ax)	(1+tan(ax)^2)a
(1/a)arctan(x/a)	1/(a^2+x^2)
ln(x)	1/x
Sinh(x)	cosh(x)
exp(-ax)sin(bx)	exp(-ax){-a sin(bx)+b cos(bx)}
exp(-ax^2)	-2axexp(-ax^2)
arccos(x)	-1/sqrt(1-$x^{2)}$

arcsin(x)	1/sqrt(1-x^2)
xsqrt(1-x^2)	(1-2x2)/sqrt(1-x2)

You can use this table to calculate, for instance, the area between an origin centered circle of radius R=1 given by $x^2+y^2=1$ and a chord y=a with a<1 . The chord cuts the circle at x=±sqrt(1-a^2). We have-

$$\text{Area}==2\int_0^{\sqrt{1-a^2}}[\sqrt{1-x^2}-a]dx=-a\sqrt{1-a^2}+\arcsin[sqrt(1-a^2)]$$

This result makes sense since the area goes to half the circle area π/2 as 'a' vanishes . Also the area is zero when a=1. One can also use a similar type of integral to predict the remaining volume in a partially filled horizontal cylindrical tank using a dip-stick as is often done at service stations.

Integrals may also be used to generate infinite series for certain functions g(x).Take the case of g(x)=ln(1-x). Here we have-

$$\ln(1-x)=-\int_1^x\frac{dx}{1-x}=-\int\sum_{n=0}^{\infty}(x^n)dx=-\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)}=-[x+\frac{x^2}{2}+\frac{x^3}{3}+...]$$

For x=-1 this reads-

$$\ln(2)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+1\text{-...}$$

This is an extremely slowly convergent series which eventually will approach 0.693147… A faster convergent series follows from-

$$\ln\left[\frac{1-x}{1+x}\right]=-\int_0^x\frac{2dx}{(1-x^2)}=-2\int_0^x\sum_{n=0}^{\infty}x^{2n}dx=-2\sum_{nm=0}^{\infty}\frac{x^{2n+1}}{2n+1}$$

Here if x=1/2 we get-

$$\ln(3)=\{1+\frac{1}{3\cdot4}+\frac{1}{5\cdot16}+\frac{1}{7\cdot64}+...\}=1.098612...$$

CONCLUDING REMARKS:

The above has condensed a years worth of calculus into a few pages giving you calculus in a nutshell. For those students about to enter your first calculus course the above should be a convenient aide as a supplement to the usual texts which can be hundreds of pages long. Remember when teaching new concepts one should always keep in mind to keep things as simple as possible. The more esoteric aspects of a topic can be taught and learned later. This approach has served me very well during my forty year career of teaching applied mathematics first at Rensselaer Polytechnic Institute and later at the University of Florida. It has led to a total of eight teaching awards over that time span including the Teacher of the Year award at the University of Florida in 1991.

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