Consider the second order PDE

\[ a(x,y)z_{xx} + 2b(x,y)z_{xy} + c(x,y)z_{yy} + F(z_{xx}, z_{yy}, z_{x}, z_{y}) = 0 \]

and introduce the characteristic variables \( \eta(x,y) \) and \( \xi(x,y) \). In terms of these variables the first partial derivatives become

\[ z_x = \eta_x \xi + \xi_x \eta_y, \quad z_y = \eta_y \xi + \xi_y \eta_x \]

A further application of the chain rule then leads to the second derivative terms

\[ z_{xx} = \eta_x^2 \xi + 2\eta_x \xi_x \eta_y + \xi_x^2 \eta_y + \eta_y^2 \xi + \xi_y \eta_x \xi \]
\[ z_{yy} = \eta_y^2 \xi + (\eta_x \xi_y + \eta_y \xi_x) \eta \xi_y + \xi_x \xi_y \eta_y + \eta_y \eta_x \xi + \xi_y \eta_x \]
\[ z_{xy} = \eta_y^2 \eta_y \xi + 2\eta_y \xi_y \eta_x + \xi_y^2 \eta_x + \eta_x^2 \eta_y + \xi_x \eta_y \xi \]

Substituting these into the above PDE yields a new equation with only a single second derivative term left after setting the coefficient multiplying the non-mixed second partial derivatives to zero. The resultant, so called, **canonical form** of our second order PDE is

\[ B(\eta, \xi) z_{\eta\xi} = G(\eta, \xi, \eta_x \xi, \eta_y \xi) + F(\eta, \xi, \eta_x, \xi_y) \]

where

\[ B(\eta, \xi) = 2a \eta_x \xi_x + 2b(\eta_x \xi_y + \eta_y \xi_x) + 2c \eta_y \xi_y \]
Here a, b, c, and F are the terms appearing in the original PDE. Note that the condition for making the other two second partial derivative terms vanish is that the characteristic curves \( \eta(x,y) = \text{constant} \) and \( \xi(x,y) = \text{constant} \) have the x and y dependent slope

\[
\frac{dy}{dx} = \pm \frac{b \pm \sqrt{b^2 - ac}}{a}
\]

Since the functions a, b, and c are assumed to be real, we can categorize the original equation by the sign of the radical. Thus the equation is termed **Hyperbolic** when \( b^2 - ac > 0 \). It is **Elliptic** when \( b^2 - ac < 0 \) and **Parabolic** when \( b^2 - ac = 0 \). Note that hyperbolic equations have two families of real characteristics while elliptic equations have no real characteristics.