USE OF RAY TRACING TO GENERATE A CAUSTIC

The other day, while drinking a cup of coffee out of a white inner glazed ceramic mug at our local coffee shop, I noticed the formation of a perfect caustic pattern produced by light rays coming in at an almost horizontal manor and reflecting off of the inner cup surface. On returning home I simulated a much larger caustic using one of my wife's frying pans. Here is the pattern we are talking about-

CAUSTIC AT THE BOTTOM OF AN EIGHT INCH FRYING PAN

Tilted pan at four foot from a 60 watt light source

We want in this article to use optical ray tracing to show how such a caustic is generated and how the caustic relates to the nephroid and a singular solution of the Clairaut differential equation.

In our simplified model we place a circle of unit radius at the origin of a Cartesian coordinate system [x,y] and have a single light ray enter parallel to the y axis in the first quadrant. This ray will hit the circle in the fourth quadrant at point $A[\sin(\theta),-\cos(\theta)]$. The angle θ represents the angle between the incoming ray and the normal to the circle which always passes through [0,0]. The reflected ray at angle 2θ to the incoming ray will cross the negative x axis at point $B[-\sin(\theta)/\{\cos(\theta)^2-\sin(\theta)^2\},0]$. We can define the reflected ray by the equation-

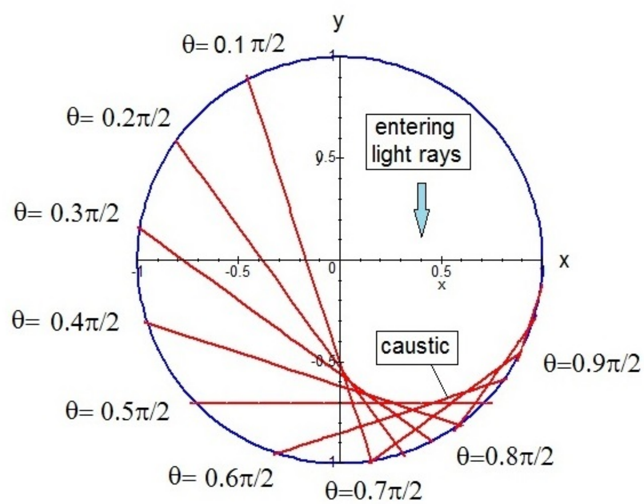
$$\alpha x+\beta y=1$$

where α and β can be evaluated using the coordinate points associated with A and B. The reflected ray equation becomes-

$$[\sin(\theta)^2 - \cos(\theta)^2]x - [\sin(2\theta)]y = \sin(\theta)$$

Here the range of θ will lie between θ=0 (incoming ray going through circle center) and θ=π/2 (incoming ray just skimming the circle at [1,0]). We next draw nine reflected rays ranging from θ=0.1π/2 through θ=0.9π/2 at equal intervals to produce the following pattern-

CAUSTIC CONSTRUCTION BY REFLECTED RAYS PRODUCED BY INCOMING LIGHT RAYS ENTERING PARALLEL TO THE Y AXIS IN THE FIRST QUADRANT



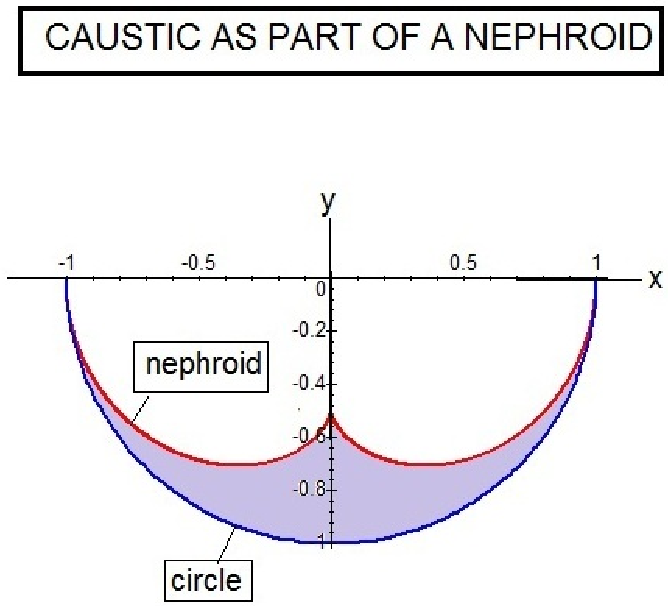
I have discarded those parts of the reflected rays lying outside of the circle. Half of a caustic is clearly shown. The second half in the third quadrant follows by letting parallel light rays enter through the second quadrant or simpler by just reflecting the caustic half pattern in the fourth quadrant about the y axis. The caustic has a cusp at [0,-05].

It is also possible to obtain the exact formula for this caustic by following the path of a point P on the periphery of a rolling disc of radius r=1/4 about a stationary inner disc of radius R=1/2. The resultant caustic pattern is known in the literature as a nephroid. In parametric form, for the present geometry, it reads-

$$x = (\frac{1}{4})\{3\cos(t) + \cos(3t)\}$$

$$y = (\frac{1}{4})\{3\sin(t) + \sin(3t)\}$$

with π <t<2π. Here is its graph-



Finally we point out that the slopes of the family straight line reflected rays as given above are equal to-

$$\frac{dy}{dx}=\frac{[\sin(\theta)^2-\cos(\theta)^2]}{\sin(2\theta)}=Const.$$

Thus the only reflected ray parallel to the x axis and hence having zero slope occurs for θ=π/4. The envelope of the collection of all reflected rays forms the caustic. This fact is very reminiscent of the solution of the Clairaut Equation-

$$y=xp+\frac{df(p)}{dp}\quad where\quad p=\frac{dy}{dx}$$

On differentiating this differential equation once with respect to x, we arrive at-

$$0=\frac{d^2y}{dx^2}\left\{x+\frac{df(p)}{dp}\right\}$$

There are two possible solutions to this last expression. The first represents all straight lines whose equation was given earlier for the caustic problem. The remaining differential expression when solved is referred to as a singular solution. This solution in the present case will be that part of a nephroid expressed in Cartesian coordinates in implicit form as-

$$[x^2+y^2-\frac{1}{4}]^3=(\frac{27}{64})x^2 \quad \text{where} \quad -1<x<+1 \quad \text{and} \quad -1<y<0\,.$$

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