FINDING THE CENTERS OF MASS AND CENTROIDS OF BODIES

If you think of the distribution of N masses m_n each located at a different position $r_n=ix_n+jy_n+kz_n$ relative to a coordinate origin, you arrive at the equalities-

$$\bar{x}\sum m_n = \sum x_n m_n \qquad \bar{y}\sum m_n = \sum y_n m_n \qquad \bar{z}\sum m_n = \sum z_n m_n$$

Here the total mass is $M = \sum m_n$ and the coordinates $[\bar{x}, \bar{y}, \bar{z}]$ give the location of the center of mass (also often referred to incorrectly as the center of gravity). A simple example of such a center of mass calculation is given by a two body configuration of mass m_1 and mass m_2 separated by a distance R. Here the mass center will be found at-

$$\bar{x} = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2} \quad where \quad R = x_2 - x_1$$

In most cases one talks about continuous mass distributions of total mass M. In that case we have from calculus that the **center of mass** is located at-

$$\bar{x} = \frac{1}{M}\iiint \rho x dxdydz \qquad \bar{y} = \frac{1}{M}\iiint \rho y dxdydz \qquad \bar{z} = \frac{1}{M}\iiint \rho z dxdydz$$

, where ρ is the variable density. If we now consider a mass of uniform density, we have that M/ρ=V the total volume. Under this condition we speak of the **centroid of a body**. Its coordinated are given by-

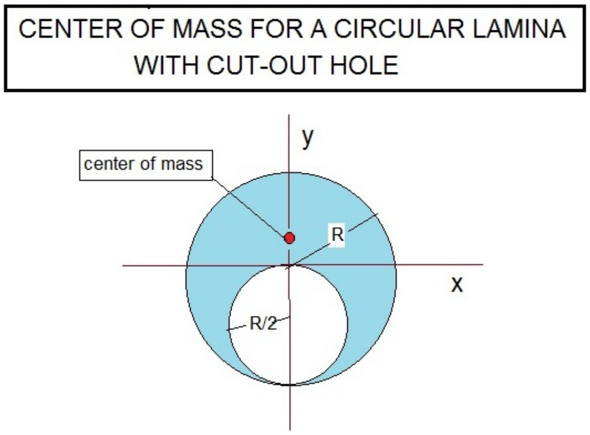
$$\bar{x} = \frac{1}{V}\iiint x dxdydz \qquad \bar{y} = \frac{1}{V}\iiint y dxdydz \qquad \bar{z} = \frac{1}{V}\iiint z dxdydz$$

The above coordinate locations can be considerably simplified in the event the mass distributions have the form of 2D laminas. Also considerations of symmetry often aid in quickly finding a solution for the center of mass.

Let us consider some specific cases.

CENTER OF MASS CALCULATIONS:

As a first example consider a 2D circular plate of radius R and of constant area density ρ. Into this plate is stamped a hole of radius R/2 centered on the negative y axis as shown-



At first glance this appears to be a difficult calculation to find $[\bar{x},\bar{y}]$. However, looking at the symmetry, treating the hole as a zero density disc, and knowing that a circular disc of constant area density has the center of mass at its center, allows us to state that-

$$\bar{y}=\frac{\rho\pi R^2\cdot 0-\rho\pi(R^2/4)\cdot(-R/2)}{\rho\pi(R^2-R^2/4)}=\frac{R}{6}$$

Hence the cm lies at [0,R/6]. One can verify this result experimentally by noting that a horizontal pin through this cm will hold the disc vertically at any angle without it rotating. This is because the net moment about this point in a constant gravity field will be zero by definition.

As a second example consider finding the center of mass of a cone of base radius R and height z=H with a variable density of $\rho=\rho_o\{1-(z/H)\}$. Here we see at once from symmetry that $\bar{x}=\bar{y}=0$. Here we also have-

$$M=\rho_o\pi R^2\int_{z=0}^{H}(1-\frac{z}{H})^3dz=\rho_o\pi R^2(\frac{H}{4})$$

So it follows that-

$$\bar{z}=[\rho_o\pi R^2/M]\int_{z=0}^{H}z(1-\frac{z}{H})^3dz=\frac{H}{5}$$

If the cone had a uniform density throughout the value of $\bar{z}$ would be H/4. So a cone with constant density will be less stable in a downward gravity field than one where the density decreases going upward. Certain toys consisting of a high density hemisphere attached to the base of a lower density cone make for interesting stand-up dolls which fascinate young pre-schoolers.

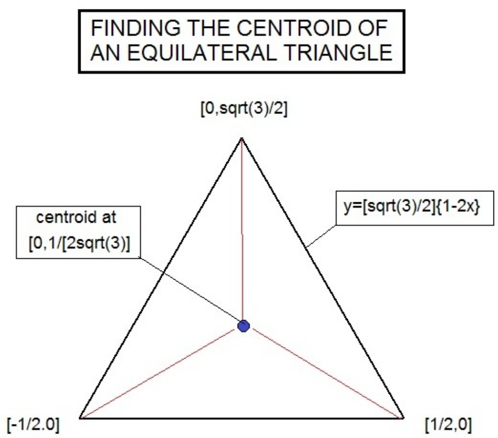
As a third example consider the center of mass of a star of radius R whose the density ρ=ρ(r) depends in radial direction only. Here the star mass is-

$$M = \iiint \rho(r)dVol = 4\pi \int_{r=0}^{R} \rho(r) r^2 dr$$

For constant density $\rho=\rho_o$, this mass reduces to $M=(4/3)\pi\rho_o R^3$ .Since each shell within the star has a mass increment of $dM=4\pi r^2\rho(r)$ and its mass center must occur at r=0 by problem symmetry, we have at once that the sum of all shells making up the star also has its mass center at r=0. So without having to specify the exact form of ρ(r) one finds that the mass center of any star with a radial dependent density has its mass center at the star center. This fact allows one the take the distance between two stars in a binary star system as the distance D between their centers.

CENTROID CALCULATIONS:

Whenever a body has uniform density one can replace mass by volume in the center of mass formulas to get the basic equations for determining its centroid. Let us run through a few examples. We start with the 2D problem of finding the centroid of an equilateral triangle as shown-



We can get the answer two different ways. The first of these is based on the problem's three fold symmetry. Drawing bisector lines from each vertex we note they intersect at its centroid $[\bar{x},\bar{y}]$, where $\bar{x}=0 \quad and \quad \bar{y}=(1/3)]sqrt(3)/2=1/(2sqrt(3)$. Using calculus we find the same answer-

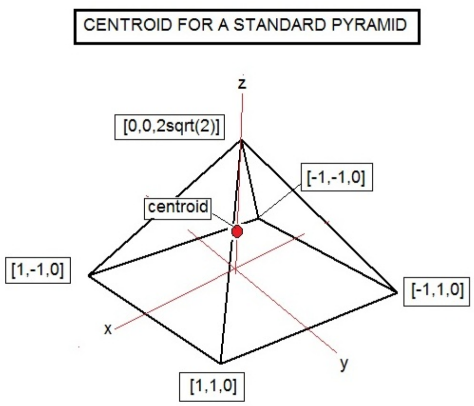
$$\bar{y}=\frac{\iint ydxdy}{\iint dxdy}=\frac{\int_{y=0}^{\sqrt{3}/2} y\{1-\frac{2y}{\sqrt{3}}\}dy}{\int_{y=0}^{\sqrt{3}/2}\{1-\frac{2y}{\sqrt{3}}\}dy}=\frac{1}{2\sqrt{3}}$$

Another centroid problem involves a constant density hemisphere of radius R.Its volume is just V=2πR^3/3, so that-

$$\bar{z}=\frac{3}{2\pi R^3}\iiint zdxdydz=\frac{3}{R^3}\int_{r=0}^{R}r^3dr\int_0^{\pi/2}\cos(\theta)\sin(\theta)d\theta[]=(\frac{3}{8})R$$

The use of spherical coordinates z=rcos(θ), y=rsin(θ)sin(φ), and x=rsin(θ)cods(φ) make the evaluation of $\bar{z}$ relatively easy.

As a third centroid calculation consider a standard pyramid as shown-



This solid structure has a square base and four equilateral triangles as sides. From symmetry we know that both $\bar{x}$ *and* $\bar{y}$ have zero value while $\bar{z}$ is given by the integral-

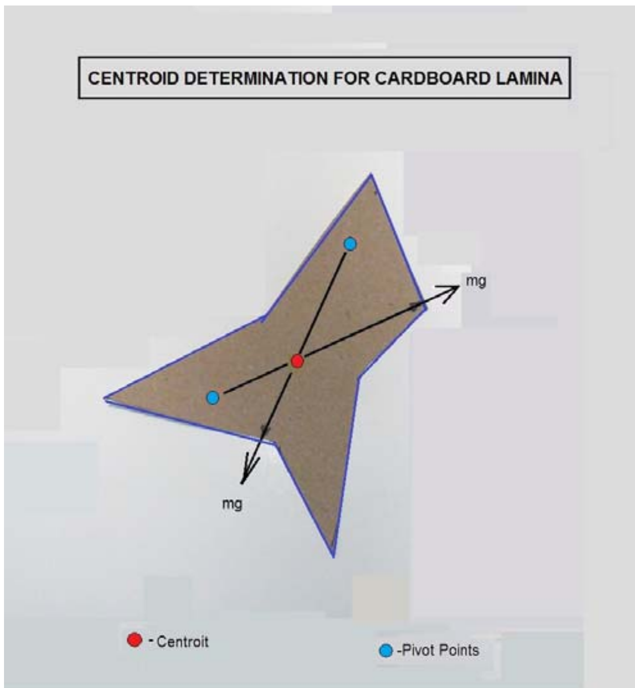
$$\bar{z}=\frac{\int_{z=0}^{\sqrt{2}}z(\sqrt{2}-z)^2dz}{\int_{z=0}^{\sqrt{2}}(\sqrt{2}-z)^2dz}=\frac{1}{2\sqrt{2}}$$

Like for a uniform density cone the centroid relative to the height is here given by-

$$\frac{\bar{z}}{H}=(\frac{1}{2\sqrt{2}})(\frac{1}{\sqrt{2}})=\frac{1}{4}$$

If one looks at the great pyramid of Cheops(alias Khufu) on the Giza plateau in Egypt, one observes the kings chamber inside lies at a point not far removed from the pyramid's centroid. Did the builders realize this?

Finally we want to point out that centroids for complex shaped 2D laminas can often be found with less effort than required by a mathematical approach. The procedure, which physics majors learn about in their elementary physics lab, consists of pivoting a given vertically orientated lamina about two different pin-holes in the lamina. Since the sum of the moments about any of the pinholes must be zero under equilibrium conditions the centroid must lie somewhere along the vertical line below the pivot point. Doing this for two different pivot points leads to two lines which cross each other at the centroid. I show you this approach using a 2D cardboard lamina below-



If one were to pivot about the point shown in red, the lamina will remain stationary no matter what vertical orientation taken.

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