ON A CLASS OF INTEGER SPIRALS

We have written several articles in recent years on a special class of integer spirals. These discussions can be found on our web pages MATHFUNC and RIC’S TECH BLOG. We want here to explore further the general mathematical properties of such point spirals and in particular show one form which beautifully separates composite from prime numbers. Our starting point is the complex function-

$$F(z, n, m) = n \exp(i \frac{n\pi}{m})$$

, where $z=x+iy$, $n$ are the running positive integers 1,2,3,4,5,… and $m$ is a fixed positive integer. The real and imaginary parts of this point function are-

$$x = n \cos\left(\frac{n\pi}{m}\right) \quad \text{and} \quad y = n \sin\left(\frac{n\pi}{m}\right)$$

In polar coordinates one has –

$$r = \sqrt{x^2 + y} = n \quad \text{and} \quad \theta = \frac{n\pi}{m}$$

These represent discrete points in the $z$ plane lying along an Archimedes spiral given by-

$$r = \left(\frac{m}{\pi}\right) \theta$$

One is now free to choose an integer value for $m$ in order to get the distribution of all positive integers along that particular integer spiral. Historically, the first of the cases we discovered and examined in detail was the $m=4$ case. This leads to the spiral shown-
The numbers n=1,2,3,4,5… have been placed at [x,y]=[ncos(nπ/4), nsin(nπ/4)]. Also we have added eight radial lines which intersect the modified Archimedes spiral at a given integer. Note that we have connected neighboring integers by straight lines since we are only interested in whole positive numbers. Those integers lying along either the x or y axis are even and are separated from each other by a factor of eight. The diagonal lines 8n+1, 8n+3, 8n+5, and 8n+7 contain all odd integers again separated from each other by factors of eight. Along which particular radial line an integer n falls is determined via the modular arithmetic operation of n mod(8) and turn of the spiral it is located at is just equal to the whole integer lying below n/8. Thus the number n=34692365 lies along the 4336545th turn of the spiral along the radial line 8n+5. This integer spiral also shows very nicely that the product of two odd numbers (8n+1)(8n+3) must always be an odd number lying along line 8n+3. Likewise the product (8n+3)(8n+5) must lie along the radial line 8n+7 since 15 mod(8)=7. The m=4 spiral also shows that the product of an even number times an odd number must always be even. So 124 x 379=46996. Without actually carrying out the product we have 124 mod(8)=4 and 379 mod(8)=3, so that (3x4) mod(8)=4. Hence the product lies along 8n+4 and is thus an even number. All prime number, except for two, are observed to lie along the diagonal lines 8n+1, 8n+3, 8n+5, and 8n+7. It would be nice to restrict the prime locations to fewer diagonal lines. Changing m might accomplish this.

Our thinking of what m to use to concentrate prime numbers in the z plane developed as follows. If we write down the first few primes greater than 3, we get the following:

<table>
<thead>
<tr>
<th>prime</th>
<th>5</th>
<th>7</th>
<th>11</th>
<th>13</th>
<th>17</th>
<th>19</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>6n±1</td>
<td>6(1)-1</td>
<td>6(1)+1</td>
<td>6(2)-1</td>
<td>6(2)+1</td>
<td>6(3)-1</td>
<td>6(3)+1</td>
<td>6(4)-1</td>
</tr>
<tr>
<td>prime</td>
<td>29</td>
<td>31</td>
<td>37</td>
<td>41</td>
<td>43</td>
<td>47</td>
<td>53</td>
</tr>
<tr>
<td>6n±1</td>
<td>6(5)-1</td>
<td>6(5)+1</td>
<td>6(6)+1</td>
<td>6(7)-1</td>
<td>6(7)+1</td>
<td>6(8)-1</td>
<td>6(9)-1</td>
</tr>
<tr>
<td>prime</td>
<td>59</td>
<td>61</td>
<td>67</td>
<td>71</td>
<td>73</td>
<td>79</td>
<td>83</td>
</tr>
<tr>
<td>6n±1</td>
<td>6(10)-1</td>
<td>6(10)+1</td>
<td>6(11)+1</td>
<td>6(12)-1</td>
<td>6(12)+1</td>
<td>6(13)+1</td>
<td>6(14)-1</td>
</tr>
</tbody>
</table>

This shows that primes greater than 3 are all of the type p=6k±1 without exception. At the same time it is noticed that not all numbers of this form are prime such as the composites 49, 55, 65, 77, etc. It does, however, allow us to make the statement:

**All primes greater than p=3 are of the form 6k±1, with k being a positive integer.**

We have found no exception to this rule going to primes of up to 20 digit length. A test for any known prime number p of the type p=6(k)±1 is that at least one of the two following possibilities is valid:

\[(p±1) \text{mod}(6)=0 \quad \text{or} \quad (p-1)\text{mod}(6)=0\]
If we take the prime number \( p = 82375104378989 \), we indeed find 
\((p+1) \mod(6) = 0\). A double prime occurs when \( p = 6n+1 \) and \( q = 6n-1 \) are both prime. Thus the combinations of \( p = 61 \) and \( q = 59 \) and \( p = 88603 \) and \( q = 88601 \) are examples of double primes.

Getting back to other integral Spirals, we recognize, in view of the top discussions, that \( m = 3 \) should make for an interesting Integer Spiral in which prime numbers will only fall on two out of six radial lines. Let us demonstrate. Setting \( m = 3 \) in the above complex function \( F(z,n,3) \) produces –

\[
[x, y] = [n \cos \left( \frac{n\pi}{3} \right), n \sin \left( \frac{n\pi}{3} \right)]
\]

Plotting the locus of points in the \( z \) plane yields the following-

Here the positive integers lie at the intersections of one of six radial lines and a hexagonal spiral as shown. The interesting result is that all \( m \) primes greater than three lie on just two diagonal curves. This means if a number is prime it must have \( n \mod(6) = 1 \) or \( n \mod(6) = 5 \). Even if it satisfies this criterion, one must still show that it is also not divisible by a smaller prime less than \( \sqrt{n} \). Thus, although \( n = 49 = 6(8) + 1 \), it cannot be a prime since \( 49/7 = 7 \). This resultant pattern for localizing primes is much more informative than that produced in a Ulam Spiral where the primes are scattered everywhere across the \( x-y \) plane. The fact that 2 and 3 do not fit the \( 6n \pm 1 \) condition suggests the possibility that at least 2
should be discarded as a standard prime number just as the number 1 was by earlier mathematicians. Since 2 is the only prime number which is even it plays somewhat the same role as the former planet Pluto does in our solar system.

Another interesting Integer Spiral corresponds to \( m=6 \). Here we get the following picture-

![Integer Spiral for \( m=6 \)](image)

Here we present the first few positive integers as the intersection points of twelve radial lines with the first four turns of a modified Archimedes (12 sided) Spiral. The picture is reminiscent of a spider web. In web construction the spider first lays the radial lines and then walks around the origin following an outward going spiral to make straight line connections between neighboring crossing points. Note that even numbers lie only along those radial lines defined by \( 12n + \text{even number} \). Odd numbers lie along lines \( 12n + \text{odd number} \). The nice compactness concerning prime numbers found for \( m=3 \) is now lost. However, one can still make certain statements such that all Mersenne Numbers defined as \( M(p)=2^p-1 \) have \( M(p) \mod (12) \) values of 7. This means they are only found along the single radial line corresponding to \( 12n+7 \). If we want to know along which radial line the product \( 39 \times 51 \times 73 \) lies, we need only perform a \( \mod (12) \) operation. This yields \( 3 \times 3 \times 1 = 9 \). So the product lies on the radial line \( 12n+9 \). That this is correct follows by noting the full product 145197 indeed has an \( n \mod (12) \) of 9.

As the last specific case we look at the integer Spiral corresponding to \( m=12 \). Here we get the following picture-
This time the straight line distance between neighboring integers becomes quite small and the modified Archimedes spiral with corners morphs into a standard Archimedes Spiral. There are a total of 24 radial lines intersecting the spiral at the integers. Each turn of the spiral contains 24 integers. Thus the number 96 will lie at the beginning of the fifth turn located at $r=96$ and angle $\theta \mod(2\pi)=0$. All Mersenne Primes greater or equal to 31 have $M(p)\mod(24)=7$. So they must lie along the radial line $24(n)+7$. The next Mersenne prime at $128-1=127$ indeed has $127\mod(24)=7$. The Fermat Numbers and Primes have the form $2^{2^k} +1$, where $k=1,2,3,4,5,…$. They read $F(1)=5$, $F(2)=17$, $F(3)=257$, and $F(4)=65537$. Except for $F(1)$ they all have $F(k)\mod(24)=17$. This means they all lie along the radial line $24(n)+17$.

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