PROPERTIES OF COMPLETE ELLIPTIC INTEGRALS

Complete Elliptic Integrals of the first and second kind are defined as-

$$K(m)=\int_{\theta=0}^{\pi/2}\frac{d\theta}{\sqrt{1-m\sin(\theta)^2}}\quad and\quad E(m)=\int_{\theta=0}^{\pi/2}\sqrt{1-m\sin(\theta)^2}\,d\theta$$

, respectively. Their complimentary forms are given as-

$$K'(m)=K(1-m)\quad and\quad E'(m)=E(1-m)$$

Typically |m|<1 and often you will see m replaced by k^2 in mathematical texts and canned programs.

Historically these integrals were first encountered in connection with the period of a simple pendulum and in the determination of the circumference of an ellipse. Their utility however extends much further and we will now look at some of their properties. Lets begin with some simple substitutions-

$$t=\sin(\theta)=u/\sqrt{u^2+1}=\tanh(w)\quad,\quad s=\sinh(w)$$

These produce the alternate forms-

$$K(m)=\int_{t=0}^{1}\frac{dt}{\sqrt{(1-t^2)(1-mt^2)}}=\int_{u=0}^{\infty}\frac{du}{\sqrt{(1+u^2)(1+(1-m)u^2)}}$$
$$=\int_{w=0}^{\infty}\frac{dw}{\sqrt{m+(1-m)\cosh(w)^2}}=\int_{s=0}^{\infty}\frac{ds}{\sqrt{(1+s^2)(1+(1-m)s^2)}}$$

and-

$$E(m)=\int_{t=0}^{1}\frac{\sqrt{1-mt^2}}{\sqrt{1-t^2}}dt=\int_{u=0}^{\infty}\frac{\sqrt{(1+u^2)(1+(1-m)u^2)}}{(1+u^2)^2}du$$
$$=\int_{w=0}^{\infty}\frac{\sqrt{(1-m)\cosh(w)^2+m}}{\cosh(w)^2}dw=\int_{s=0}^{\infty}\sqrt{\frac{1+(1-m)s^2}{(s^2+1)^3}}ds$$

We can expand the above integral in t for K(m) to find-

$$K(m)=\int_{t=0}^{1}\frac{dt}{\sqrt{1-t^2}}[1+\frac{1}{2}mt^2+\frac{(1\cdot 3)}{(2\cdot 4)}(mt^2)^2+\frac{(1\cdot 3\cdot 5)}{2\cdot 4\cdot 6)}(mt^2)^3+...]$$

This result can be integrated term by term to yield the well known result-

$$K(m)=\frac{\pi}{2}[1+\left(\frac{1}{2}\right)^2 m+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 m^2+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 m^3+...]=\frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{4^n(n!)^2}\right]^2 m^n$$

which converges relatively rapidly to yield K(0.5) accurate to 33 places when using the first hundred terms in the series. Expanding the above t integral for E(m) yields-

$$E(m)=\int_{t=0}^{1}\frac{dt}{\sqrt{1-t^2}}[1-\left(\frac{1}{2}\right)mt^2-\left(\frac{1}{8}\right)(mt^2)^2-\left(\frac{1}{16}\right)(mt^2)^3-...]$$

and produces the infinite series-

$$E(m)=\frac{\pi}{2}\left[1-\left(\frac{1}{2}\right)^2 m-\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{m}{3}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{m^2}{5}-....\right]$$

One can also use the above integral representations for the Complete Elliptic Integrals to obtain the following evaluations-

$$\sqrt{2}\int_{0}^{\infty}\frac{ds}{\sqrt{(1+s^2)(2+s^2)}}=K(0.5)=1.854074677301371918433850334720...$$

and-

$$\frac{1}{\sqrt{2}}\int_{0}^{\infty}\frac{\sqrt{2+s^2}}{\sqrt{(1+s^2)^3}}ds=E(0.5)=1.3506438810476755025201747353 4...$$

Numerous other integrals can be expressed in terms of K(m) and E(m). The reader is referred to the "Handbook of Mathematical Functions" by Abramowitz and Stegun.

One can also obtain derivative conditions using the Leibnitz approach. For example,

$$\frac{dE(m)}{dm}=\frac{1}{2m}\int_0^1\frac{[(1-mt^2)-1]}{\sqrt{1-t^2}\sqrt{1-mt^2}}dt=\frac{1}{2m}[E(m)-K(m)]$$

and-

$$\frac{dK(m)}{dm}=[\frac{E(m)-(1-m)K(m)}{2m(1-m)}]$$

Another important property of the complete elliptic integrals is the Legendre observation that-

$$\left[K(m)E(1-m)+K(1-m)E(m)-K(m)K(1-m)\right]=\frac{\pi}{2}$$

which at m=0.5 produces the very useful identity-

$$\pi=2K(0.5)\left[2E(0.5)-K(0.5)\right]]$$

Also, in view of the derivative condition on K(m) given above, it follows that-

$$\pi=2K(0.5)\frac{dK(0.5)}{dm}=\left|\quad\frac{d[K(m)]^2}{dm}\right|_{m=1/2}=3.14159265358...$$

That is, π is given exactly as the product of two integrals as-

$$\pi=4\left[\int_{s=0}^{\infty}\frac{ds}{\sqrt{2+3s^2+s^4}}\right]\left[\int_{s=0}^{\infty}\frac{s^2}{\sqrt{(1+s^2)(2+s^2)^3}}ds\right]$$

$$=4\left[\int_{t=0}^{1}\frac{dt}{\sqrt{(1-t^2)(2-t^{2)}}}\right]\left[\int_{t=0}^{1}\sqrt{\frac{1-t^2}{2-t^2}}dt\right]$$

$=4(1.311028777146059905232419794945559706841377475715811581408410851900395293535207 1..)(0.59907011736779610371996124614016193911360633160782577913183747647320260707195 79...)=3.1415926535897932384626433832795028841971693993751058209749445923078164062862 09..$

yielding π to eighty places.

These relations between π and K(0.5) and E(0.5), when used in conjunction with the AGM method for obtaining very precise values for these complete elliptic integrals , allows one to determine the value of π to billion place accuracies(see- http://numbers.computation.free.fr/Constants/Pi/piAGM.html) . Since the last integrals in t can be solved exactly in terms of gamma functions Γ, one finds the identity-

$$K(0.5)=\frac{\sqrt{\pi}\,\Gamma(1/4)}{2^{3/2}\Gamma(3/4)}=\frac{\Gamma(1/4)^2}{4\sqrt{\pi}}=\frac{\pi}{\sqrt{2}M}=1.8540746773013719184..$$

where M=1.1981402347355922075.. is the algebraic-geometrical mean(AGM) of 1 and √2. We thus have-

$$\pi=\left[\frac{M\,\Gamma^2(1/4)}{2\sqrt{2}}\right]^{2/3}=\frac{MB(0.25,0.25)}{2\sqrt{2}}=\frac{M}{2\sqrt{2}}\int_{t=0}^{1}\frac{dt}{[t(1-t)]^{3/4}}$$

for determining the value of π to any desired order of accuracy. Here we have used the Beta function relations-

$$B(n,m)=\frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}=\int_{t=0}^{1}t^{n-1}(1-t)^{m-1}\,dt$$

Finally, let us formulate some differential equations involving K(m) and E(m). Consider first the hypergeometric series-

$$F(a,b,c,m)=1+\frac{ab}{c}m+\frac{a(a+1)b(b+1)}{c(c+1)2!}m^2+\frac{a(a+1)(a+2)b(b+1(b+2)}{c(c+1)(c+2)3!}m^3+...$$

If we set a=b=1/2 and c=1, one obtains-

$$F(1/2,1/2,\,1,\,m)=1+\frac{1}{4}m+\frac{9}{64}m^2+\frac{25}{256}m^3+O(m^4\}$$

This just matches 2K(m)/π from the earlier given series expansion. Thus we can conclude, knowing the form of the standard hypergeometric equation, that-

$$m(1-m)\frac{d^2K(m)}{dm^2}+(1-2m)\frac{dK(m)}{dm}-\frac{1}{4}K(m)=0$$

Also, by manipulating the two first derivative conditions for K(m) and E(m) given above, we find the second order differential equation-

$$m(1-m)\frac{d^2E(m)}{dm^2}+(1-m)\frac{dE(m)}{dm}+(\frac{1}{4})E(m)=0 \quad .$$

This is just another version of the hypergeometric equation with a=1/2, b=-1/2, and c=1. Indeed one has that-

$$E(m)=\frac{\pi}{2}F(\frac{1}{2},\frac{-1}{2},1,m) \quad \textit{provided that } |m|<1 \text{ .}$$

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